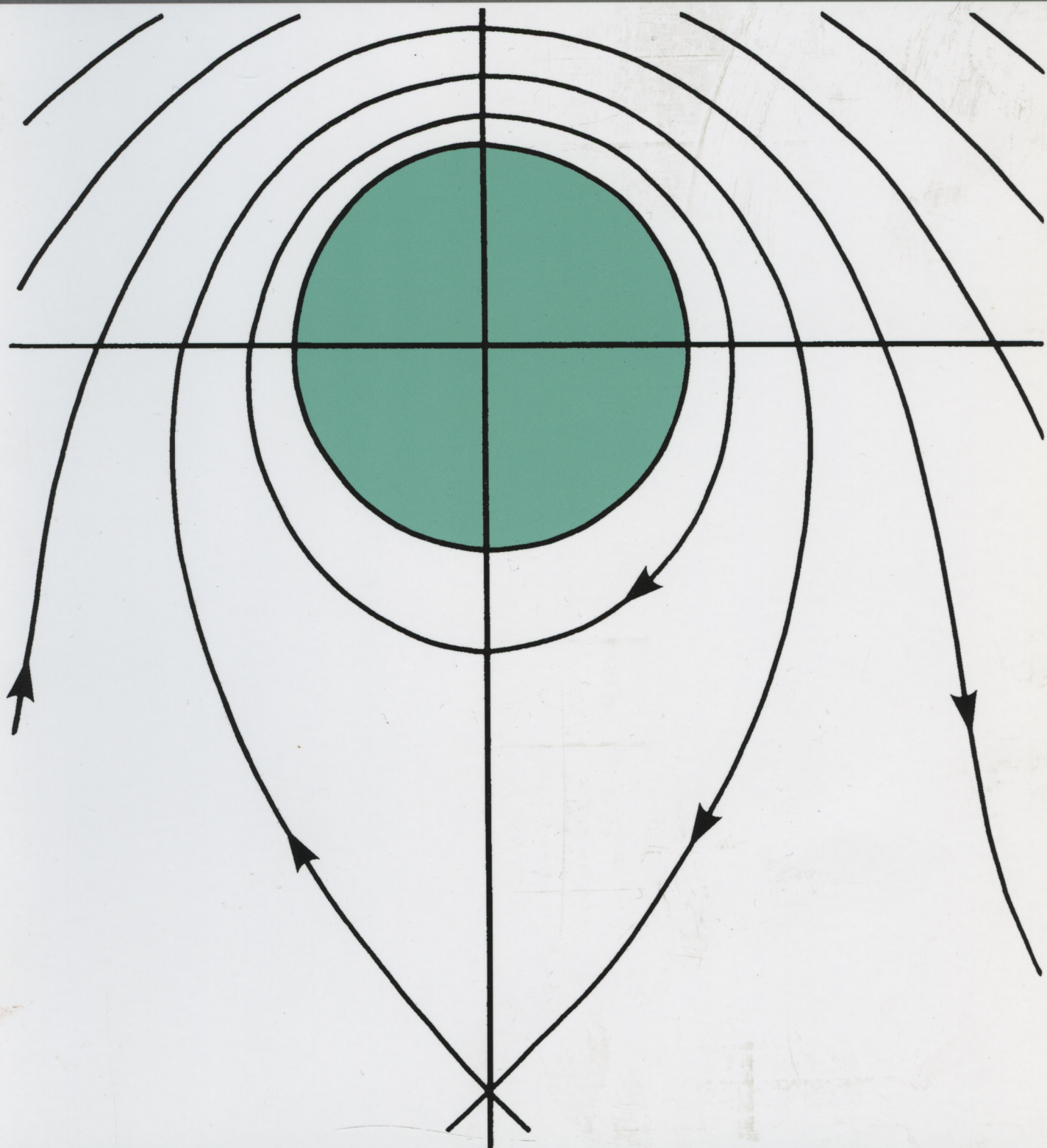


# COMPLEX ANALYSIS

## UNIT D3 THE MANDELBROT SET





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## UNIT D3 THE MANDELBROT SET

*Prepared by the Course Team*

The Open University, Walton Hall, Milton Keynes, MK1 6AA  
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Before working through this text, make sure that you have read the  
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# INTRODUCTION

Many techniques for solving equations involve *iteration*, otherwise known as the method of 'refining guesses'. For example, one way to solve a real polynomial equation  $p(x) = 0$  is to use the Newton–Raphson method, based on the recurrence relation:

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}, \quad n = 0, 1, 2, \dots$$

The justification of this recurrence relation is shown in Figure 0.1, which indicates that if  $x_n$  is close to a zero  $a$  of  $p$ , then  $x_{n+1}$  is (usually) even closer. In particular, if  $p(x) = x^2 - 2$ , then the Newton–Raphson recurrence relation is

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2} \left( x_n + \frac{2}{x_n} \right), \quad n = 0, 1, 2, \dots$$

In this unit we study the behaviour of *complex* iterative processes of the form

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \dots,$$

where  $f$  is an analytic function. Usually  $f$  will be a polynomial function, although we spend a short time in Section 1 discussing the Newton–Raphson method for complex polynomials, which involves the iteration of a rational function. Most of Section 1 is devoted to the basic notions associated with iteration: *nth iterates*, *fixed points* and *conjugate iteration sequences*.

Section 2 takes up the iteration of complex quadratic functions, which is the main topic of the unit. We begin by showing that it is sufficient to consider quadratic functions belonging to the basic family  $\{P_c(z) = z^2 + c : c \in \mathbb{C}\}$ , because each quadratic function is equivalent, in a certain sense, to one of this form. We then consider, for any given  $c \in \mathbb{C}$ , the set of points  $E_c$  which 'escape to  $\infty$ ' under iteration of  $P_c$ , and establish some basic properties of  $E_c$  and its complement  $K_c$ . To determine some of the points of  $K_c$ , we introduce the idea of a *periodic cycle* of points, and we show that certain periodic cycles must lie in the interior of  $K_c$ , whereas others lie on the boundary of  $K_c$ . This boundary is called the *Julia set*  $J_c$  of  $P_c$  (see Figure 0.2).

In Section 3, we use a technique called *graphical iteration*, which applies only to real functions, to obtain various properties of the set  $K_c$  when  $c$  is real. For example, we determine the nature of the set  $K_c \cap \mathbb{R}$  for all real numbers  $c$ .

In Section 4, we define the *Mandelbrot set*, which is the set  $M$  of numbers  $c$  such that the set  $K_c$  is connected, and we obtain a criterion for  $c$  to belong to  $M$ . This criterion makes it possible to plot pictures of  $M$  (see Figure 0.3) and thus reveal its immensely complicated structure. We investigate this structure by using a result which states that if  $P_c$  has a so-called *attracting cycle*, then  $c \in M$ .

You may be tempted to think that the Mandelbrot set is in some sense an oddity, arising from some special property of quadratic functions. However, in Section 5, we describe briefly some different iteration sequences, and we find that the Mandelbrot set is a truly universal object.

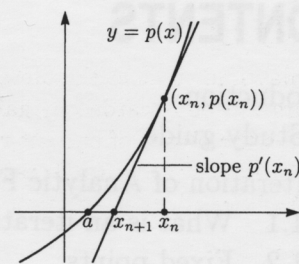


Figure 0.1

$$p'(x_n) = \frac{p(x_n)}{x_n - x_{n+1}}$$

An initial guess of  $x_0 = 2$  yields the sequence of refined guesses 2, 1.5, 1.417, ..., which converges rapidly to the known solution  $\sqrt{2}$ .

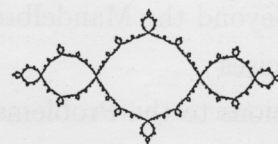


Figure 0.2 A Julia set

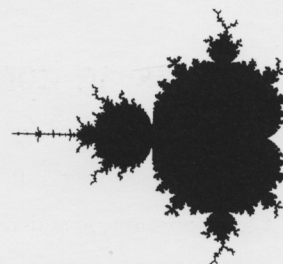


Figure 0.3  
The Mandelbrot set



## Study guide

You may find some of the ideas in this unit quite challenging. However, your labours will be rewarded by gaining some insight into a fascinating subject, which is the object of current research. Some of this research is mentioned in the discussion on the audio tape, which has no associated frames and which may be listened to at any time.

Section 5 and Subsection 2.4 are intended for reading only.

Associated with this unit is a segment of the Video Tape for the course. Although this unit text is self-contained, access to the video tape will enhance your understanding. Suitable points at which to view the video tape are indicated by a symbol placed in the margin.

# 1 ITERATION OF ANALYTIC FUNCTIONS

After working through this section, you should be able to:

- (a) calculate several terms of a given *iteration sequence*;
- (b) determine the  $n$ th *iterate* of a given analytic function, for small values of  $n$ ;
- (c) determine the *fixed points* of certain analytic functions and find their nature;
- (d) calculate *conjugate iteration sequences*;
- (e) determine the Newton–Raphson function  $N$  for a given polynomial function  $p$ , and describe the iterative behaviour of  $N$  when  $p$  is a quadratic function.

## 1.1 What is an iteration sequence?

Any sequence  $\{z_n\}$  defined by a recurrence relation of the form

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \dots,$$

where  $f$  is a function, is called an **iteration sequence** with **initial term**  $z_0$ .

For example, if  $f(z) = 2z$ , then the recurrence relation is

$$z_{n+1} = 2z_n, \quad n = 0, 1, 2, \dots$$

If  $z_0 = i$ , then the first few terms of the corresponding iteration sequence are

$$z_0 = i, \quad z_1 = 2i, \quad z_2 = 4i, \quad z_3 = 8i, \dots$$

In this case the sequence  $\{z_n\}$  tends to infinity, but if we had chosen  $z_0 = 0$ , then the corresponding iteration sequence would be constant:

$$z_0 = 0, \quad z_1 = 0, \quad z_2 = 0, \quad z_3 = 0, \dots$$

Thus the behaviour of a given iteration sequence depends not only on the function  $f$ , but also on the choice of initial term  $z_0$ . We often represent such iteration sequences, as in Figure 1.1, by plotting the points  $z_0, z_1, \dots$ , and indicating how these points are related by the function  $f$ . Note that, in effect, we are plotting the domain and codomain of  $f$  on the same diagram!

Some texts call  $z_0$  the *seed* and  $\{z_n\}$  the *orbit* of  $z_0$ . Also note that the study of iteration sequences is often called *dynamical systems*.

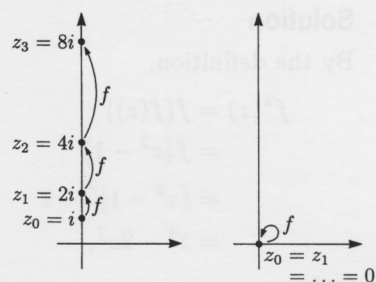


Figure 1.1

## Problem 1.1

Calculate and plot the terms up to  $z_3$  of each of the following iteration sequences. Also, write down the corresponding functions  $f$ .

- (a)  $z_{n+1} = z_n^2$ ,  $z_0 = i$     (b)  $z_{n+1} = \frac{1}{2}z_n + 1$ ,  $z_0 = 0$   
 (c)  $z_{n+1} = z_n^2 - 1$ ,  $z_0 = 0$     (d)  $z_{n+1} = z_n^2 + i$ ,  $z_0 = 0$

In order to study iteration sequences systematically, it is useful to introduce various basic notations. For example, if the sequence  $\{z_n\}$  is defined by

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \dots,$$

then

$$z_1 = f(z_0), \quad z_2 = f(z_1) = f(f(z_0)), \quad z_3 = f(z_2) = f(f(f(z_0))),$$

and, in general,

$$z_n = f(f(\dots(f(z_0))\dots)), \quad \text{for } n = 1, 2, \dots, \quad (1.1)$$

where the function  $f$  is applied  $n$  times. We introduce a notation for such repeated compositions.

**Definition** The  $n$ th iterate of a function  $f$  is the function obtained by applying the function  $f$  exactly  $n$  times:

$$f^n = f \circ f \circ \dots \circ f.$$

Also,  $f^0$  denotes the identity function  $f^0(z) = z$ .

For example,

$$f^1(z) = f(z)$$

and

$$f^2(z) = f(f(z)).$$

## Remarks

- 1 There is, of course, a possible confusion between

$$f^2 = f \circ f \quad \text{and} \quad f^2 = f \times f,$$

but the context will make the intended meaning clear.

- 2 Equation (1.1), relating the general term  $z_n$  to the initial term  $z_0$ , can now be written in the convenient form

$$z_n = f^n(z_0), \quad n = 1, 2, \dots;$$

see Figure 1.2.

- 3 Note that if  $m, n \geq 0$ , then

$$f^m(f^n(z)) = f^{m+n}(z) = f^n(f^m(z)),$$

since composition of functions is associative.

## Example 1.1

Determine the rules for the functions  $f^2$  and  $f^3$  when  $f(z) = z^2 - 1$ .

## Solution

By the definition,

$$\begin{aligned} f^2(z) &= f(f(z)) \\ &= f(z^2 - 1) \\ &= (z^2 - 1)^2 - 1 \\ &= z^4 - 2z^2, \end{aligned}$$

and so

$$\begin{aligned} f^3(z) &= f(f^2(z)) \\ &= (z^4 - 2z^2)^2 - 1 \\ &= z^8 - 4z^6 + 4z^4 - 1. \quad \blacksquare \end{aligned}$$

There should be no confusion with the  $n$ th derivative  $f^{(n)}$ .

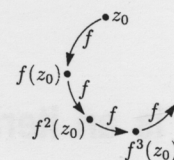


Figure 1.2

Alternatively,

$$\begin{aligned} f^3(z) &= f^2(f(z)) \\ &= (z^2 - 1)^4 - 2(z^2 - 1)^2 \\ &= z^8 - 4z^6 + 4z^4 - 1. \end{aligned}$$



### Problem 1.2

Determine the rules for the functions  $f^2$  and  $f^3$  when  $f(z) = \frac{1}{2}z + 1$ .

The solution to Example 1.1 suggests that for some functions  $f$  it may be difficult to find a general formula for the  $n$ th iterate,  $f^n$ . However, there are a few simple cases which are very useful.

### Example 1.2

Find a formula for the  $n$ th iterate  $f^n$  of each of the following functions.

- (a)  $f(z) = az$ , where  $a \in \mathbb{C}$       (b)  $f(z) = z^2$

#### Solution

- (a) By definition,

$$f^1(z) = f(z) = az,$$

$$f^2(z) = f(f(z)) = f(az) = a(az) = a^2z,$$

$$f^3(z) = f(f^2(z)) = f(a^2z) = a(a^2z) = a^3z,$$

and, in general,

$$f^n(z) = a^n z, \quad \text{for } n = 1, 2, \dots$$

This can be proved by  
Mathematical Induction.

- (b) By definition,

$$f^1(z) = f(z) = z^2,$$

$$f^2(z) = f(f(z)) = f(z^2) = (z^2)^2 = z^4,$$

$$f^3(z) = f(f^2(z)) = f(z^4) = (z^4)^2 = z^8,$$

and, in general,

$$f^n(z) = z^{2^n}, \quad \text{for } n = 1, 2, \dots \quad \blacksquare$$

The following problem deals with other  $n$ th iterates which can be found explicitly.

### Problem 1.3

Find a formula for the  $n$ th iterate  $f^n$  of each of the following functions.

- (a)  $f(z) = z + b$ , where  $b \in \mathbb{C}$

- (b)  $f(z) = z^3$

The formulas obtained in Example 1.2 and Problem 1.3 can be used to determine the behaviour of the corresponding iteration sequences, as we now illustrate.

### Example 1.3

(a) Prove that if  $f(z) = az$ , where  $|a| < 1$ , and  $z_0 \in \mathbb{C}$ , then

$$z_n = f^n(z_0) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(b) Prove that if  $f(z) = z^2$  and  $|z_0| < 1$ , then

$$z_n = f^n(z_0) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(c) Prove that if  $f(z) = z + b$ , where  $b \neq 0$ , and  $z_0 \in \mathbb{C}$ , then

$$z_n = f^n(z_0) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

### Solution

(a) By Example 1.2(a),

$$z_n = f^n(z_0) = a^n z_0, \quad \text{for } n = 1, 2, \dots$$

Since  $|a| < 1$ , we deduce that  $\{z_n\}$  is a null sequence, as required.

Unit A3, Theorems 1.2 and 1.3

(b) By Example 1.2(b),

$$z_n = f^n(z_0) = z_0^{2^n}, \quad \text{for } n = 1, 2, \dots$$

Now we use the facts that  $|z_0| < 1$  and  $2^n \geq n$ , for  $n = 1, 2, \dots$ , to deduce that

$$|z_n| = |z_0|^{2^n} \leq |z_0|^n, \quad \text{for } n = 1, 2, \dots$$

Since  $|z_0| < 1$ ,  $\{|z_0|^n\}$  is a null sequence and so, therefore, is  $\{z_n\}$ , by the Squeeze Rule.

By the Binomial Theorem,

$$\begin{aligned} 2^n &= (1+1)^n \\ &= 1 + n + \dots \geq n. \end{aligned}$$

Unit A3, Theorem 1.1

(c) By Problem 1.3(a),

$$z_n = f^n(z_0) = z_0 + nb, \quad \text{for } n = 1, 2, \dots$$

Thus

$$\begin{aligned} \frac{1}{z_n} &= \frac{1}{z_0 + nb} \\ &= \frac{1/n}{z_0/n + b} \\ &\rightarrow \frac{0}{b} = 0 \text{ as } n \rightarrow \infty \text{ (since } b \neq 0); \end{aligned}$$

so, by the Reciprocal Rule,

$$z_n \rightarrow \infty \text{ as } n \rightarrow \infty. \quad \blacksquare$$

Unit A3, Theorem 1.5

**Remark** The sequence  $\{z_0^{2^n}\}$  in part (b) tends to 0 extremely quickly if  $|z_0| < 1$ . In fact, for any  $\alpha$  such that  $0 < |\alpha| < 1$ , we have

$$\frac{|z_0|^{2^n}}{\alpha^n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus  $\{z_0^{2^n}\}$  tends to 0 faster than  $\{\alpha^n\}$  for any  $\alpha$  with  $0 < |\alpha| < 1$ .

In Example 1.3(a) we saw that  $f^n(z_0) \rightarrow 0$  as  $n \rightarrow \infty$  for all choices of initial term  $z_0$ , whereas in Example 1.3(b),  $f^n(z_0) \rightarrow 0$  as  $n \rightarrow \infty$ , whenever  $|z_0| < 1$ . In the next problem we ask you to investigate what happens for this latter function  $f$  with other initial values.

### Problem 1.4

With  $f(z) = z^2$ , determine the behaviour of the iteration sequence

$$z_n = f^n(z_0), \quad n = 1, 2, \dots,$$

when

$$(a) z_0 = 1; \quad (b) z_0 = -i; \quad (c) z_0 = e^{2\pi i/3}; \quad (d) |z_0| > 1.$$

To prove this, note that if

$$a_n = |z_0|^{2^n} / |\alpha|^n, \quad n = 0, 1, 2, \dots,$$

then

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{|z_0|^{2^{n+1}}}{|\alpha|^{n+1}} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence  $\{a_n\}$  is null, by Unit B3, Theorems 1.9 and 1.7, for example.



## 1.2 Fixed points

Whenever an iteration sequence, defined by a *continuous* function  $f$ , converges to a limit  $\alpha$ , say, then the point  $\alpha$  has the property that  $f(\alpha) = \alpha$ , as we now show. Suppose that the iteration sequence  $z_n = f^n(z_0)$  is such that

$$z_n \rightarrow \alpha \text{ as } n \rightarrow \infty.$$

Then  $z_{n+1} \rightarrow \alpha$  as  $n \rightarrow \infty$ , so that

$$\alpha = \lim_{n \rightarrow \infty} z_{n+1} = \lim_{n \rightarrow \infty} f(z_n) = f(\alpha),$$

because the function  $f$  is continuous at  $\alpha$ . As the limit  $\alpha$  satisfies  $f(\alpha) = \alpha$ , it is called a *fixed point* of the function  $f$  (see Figure 1.3).

**Definition** A point  $\alpha$  is a **fixed point** of a function  $f$  if  $f(\alpha) = \alpha$ .

For example, the function  $f(z) = 2z$  has 0 as a fixed point, whereas the function  $f(z) = z^2$  has 0 and 1 as fixed points. In general, we find the fixed points (if any) of a given function  $f$  by solving the **fixed point equation**  $f(z) = z$ .

### Problem 1.5

Determine the fixed points of each of the following functions  $f$ .

- (a)  $f(z) = \frac{1}{2}z + 1$       (b)  $f(z) = z^2 - 2$       (c)  $f(z) = z^3$

The behaviour of an iteration sequence  $z_n = f^n(z_0)$ ,  $n = 1, 2, \dots$ , near a fixed point  $\alpha$  of an *analytic* function  $f$  depends to a very great extent on the derivative of  $f$  at  $\alpha$ . If  $|f'(\alpha)| < 1$ , then, to a good approximation,  $f$  maps small discs with centre  $\alpha$  to even smaller discs with centre  $\alpha$ . Thus an initial term  $z_0$  near  $\alpha$  gives rise to an iteration sequence which is attracted to  $\alpha$  (see Figure 1.4).

**Theorem 1.1** Let  $\alpha$  be a fixed point of an analytic function  $f$  and suppose that  $|f'(\alpha)| < 1$ . Then there exists  $r > 0$  such that

$$\lim_{n \rightarrow \infty} f^n(z_0) = \alpha, \quad \text{for } |z_0 - \alpha| < r.$$

**Proof** We first choose a real number  $a$  such that  $|f'(\alpha)| < a < 1$ . Since

$$f'(\alpha) = \lim_{z \rightarrow \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} \quad \text{and} \quad a - |f'(\alpha)| > 0,$$

there is a positive number  $r$  such that

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} - f'(\alpha) \right| < a - |f'(\alpha)|, \quad \text{for } 0 < |z - \alpha| < r;$$

see Figure 1.5.

Hence

$$\left| \frac{f(z) - f(\alpha)}{z - \alpha} \right| < a, \quad \text{for } 0 < |z - \alpha| < r,$$

and so, since  $f(\alpha) = \alpha$ ,

$$|f(z) - \alpha| \leq a|z - \alpha|, \quad \text{for } |z - \alpha| < r.$$

Thus if  $|z_0 - \alpha| < r$ , then

$$|f(z_0) - \alpha| \leq a|z_0 - \alpha|,$$

so that  $|f(z_0) - \alpha| < r$ , also.

The sequence  $\{z_{n+1}\}$  is just the sequence  $\{z_n\}$  with its first term removed.

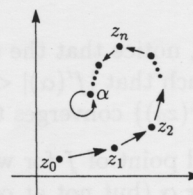


Figure 1.3

Recall from Unit A4, Subsection 1.5, that  $f'(\alpha)$  acts as a complex scale factor at  $\alpha$ .

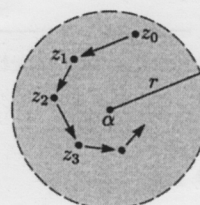


Figure 1.4

Apply the  $\varepsilon$ - $\delta$  definition of limit (Unit A3, Section 3) with

$\varepsilon = a - |f'(\alpha)|$   
and put  $r = \delta$ .

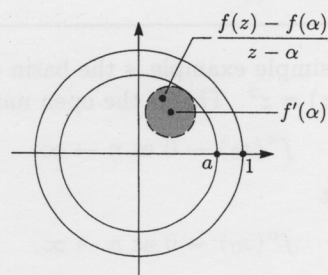


Figure 1.5

Hence

$$|f^2(z_0) - \alpha| \leq a|f(z_0) - \alpha| \leq a^2|z_0 - \alpha|,$$

and, in general,

$$|f^n(z_0) - \alpha| \leq a^n|z_0 - \alpha|, \quad \text{for } n = 1, 2, \dots \quad (1.2)$$

Since  $0 < a < 1$ , the sequence  $\{a^n\}$  is a null sequence. Thus, for  $|z_0 - \alpha| < r$ , the sequence  $\{f^n(z_0) - \alpha\}$  is null, by the Squeeze Rule, and so

$$\lim_{n \rightarrow \infty} f^n(z_0) = \alpha. \quad \blacksquare$$

In this proof, notice that the smaller  $|f'(\alpha)|$  is, the smaller we can choose the number  $a$  (such that  $|f'(\alpha)| < a < 1$ ) and so, by Inequality (1.2), the faster the sequence  $\{f^n(z_0)\}$  converges to  $\alpha$ . This convergence is very fast if  $f'(\alpha) = 0$ .

If  $\alpha$  is a fixed point of  $f$  for which  $|f'(\alpha)| > 1$ , then we should expect initial terms  $z_0$  near  $\alpha$  (but not at  $\alpha$ ) to be pushed away from  $\alpha$  by  $f$ . If  $|f'(\alpha)| = 1$ , then the behaviour depends in a more subtle way on the precise value of  $f'(\alpha)$ . These observations suggest the following classification of fixed points.

**Definitions** The fixed point  $\alpha$  of an analytic function  $f$  is

- (a) **attracting**, if  $|f'(\alpha)| < 1$ ;
- (b) **repelling**, if  $|f'(\alpha)| > 1$ ;
- (c) **indifferent**, if  $|f'(\alpha)| = 1$ ;
- (d) **super-attracting**, if  $f'(\alpha) = 0$ .

Some texts use

- (a) *attractive* or *stable*;
- (b) *repulsive* or *unstable*;
- (c) *neutral*.

Note that (d) is a special case of (a).

For example, the function  $f(z) = az$ , where  $a \in \mathbb{C}$ , has 0 as a fixed point and, since  $f'(z) = a$ , this fixed point is attracting if  $|a| < 1$ , repelling if  $|a| > 1$  and indifferent if  $|a| = 1$ .

## Problem 1.6

For each of the following functions  $f$ , classify the given fixed point  $\alpha$ .

- (a)  $f(z) = z^2$ ,  $\alpha = 0, 1$
- (b)  $f(z) = \frac{1}{2}z + 1$ ,  $\alpha = 2$
- (c)  $f(z) = z^2 - 2$ ,  $\alpha = 2$

For any given function  $f$  with an attracting fixed point  $\alpha$ , it is natural to ask exactly which points  $z$  are attracted to  $\alpha$  under iteration of  $f$  (that is,  $f^n(z) \rightarrow \alpha$  as  $n \rightarrow \infty$ ) and so we make the following definition.

**Definition** If  $\alpha$  is an attracting fixed point of an analytic function  $f$ , then the **basin of attraction** of  $\alpha$  under  $f$  is the set

$$\{z : f^n(z) \rightarrow \alpha \text{ as } n \rightarrow \infty\}.$$

Here, and subsequently, we may use  $z$ , rather than  $z_0$ , as an initial term, when we do not need to label the sequence  $\{z_n\}$ .

A simple example is the basin of attraction of  $\alpha = 0$  under the function  $f(z) = z^2$ . This is the open unit disc because

$$f^n(z_0) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad \text{for } |z_0| < 1,$$

but

$$f^n(z_0) \not\rightarrow 0 \text{ as } n \rightarrow \infty, \quad \text{for } |z_0| \geq 1 \quad (\text{since } |f^n(z_0)| \geq 1, \text{ for } n = 1, 2, \dots).$$

Recall from Example 1.3(b) that

$$f^n(z_0) = z_0^{2^n}, \quad \text{for } n = 0, 1, 2, \dots$$

Later in the unit we see some more complicated examples.



### Problem 1.7

For each of the following functions  $f$ , determine the basin of attraction of the given fixed point  $\alpha$ .

(a)  $f(z) = \frac{1}{2}z$ ,  $\alpha = 0$       (b)  $f(z) = z^3$ ,  $\alpha = 0$ .

## 1.3 Conjugate iteration sequences

Consider the iteration sequence

$$z_{n+1} = z_n^2 + 2z_n, \quad n = 0, 1, 2, \dots, \text{ with } z_0 = -\frac{1}{2}, \quad (1.3)$$

and suppose that we wish to find a formula for  $z_n$  in terms of  $n$ . This iteration sequence is quite complicated, and so it is sensible to look at the terms of  $\{z_n\}$ :

$$z_0 = -\frac{1}{2}, \quad z_1 = -\frac{3}{4}, \quad z_2 = -\frac{15}{16}, \quad z_3 = -\frac{255}{256}, \dots$$

These terms suggest that  $z_n \rightarrow -1$  as  $n \rightarrow \infty$ , and so we make the substitution

$$z_n = w_n - 1,$$

and try to find a formula for  $w_n$ . Substituting for  $z_n$  and  $z_{n+1}$  in (1.3), we obtain

$$\begin{aligned} w_{n+1} - 1 &= (w_n - 1)^2 + 2(w_n - 1) \\ &= w_n^2 - 2w_n + 1 + 2w_n - 2 \\ &= w_n^2 - 1; \end{aligned}$$

hence

$$w_{n+1} = w_n^2.$$

The iteration sequence

$$w_{n+1} = w_n^2, \quad n = 0, 1, 2, \dots, \text{ with } w_0 = z_0 + 1 = \frac{1}{2}, \quad (1.4)$$

is simpler than one given in (1.3) and, moreover, we know that

$$w_n = \left(\frac{1}{2}\right)^{2^n}, \quad \text{for } n = 0, 1, 2, \dots$$

We deduce that the formula for  $z_n$  in terms of  $n$  is

$$z_n = \left(\frac{1}{2}\right)^{2^n} - 1, \quad \text{for } n = 0, 1, 2, \dots$$

More generally, suppose that

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \dots,$$

is a given, but complicated, iteration sequence that we wish to investigate, and that  $h$  is a one-one function. Then

$$w_n = h(z_n), \quad n = 0, 1, 2, \dots,$$

is also an iteration sequence. Indeed, since  $z_n = h^{-1}(w_n)$ , we have

$$\begin{aligned} w_{n+1} &= h(z_{n+1}) \\ &= h(f(z_n)) \\ &= h(f(h^{-1}(w_n))), \quad \text{for } n = 0, 1, 2, \dots \end{aligned}$$

Thus

$$w_{n+1} = g(w_n), \quad \text{for } n = 0, 1, 2, \dots,$$

where the function  $g$  is given by  $g = h \circ f \circ h^{-1}$  (see Figure 1.6). With a suitable choice of  $h$ , the function  $g$  will be simpler than  $f$  (as in the above example, where  $f(z) = z^2 + 2z$ ,  $g(w) = w^2$  and  $h(z) = z + 1$ ).

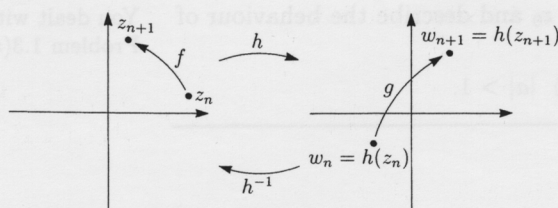


Figure 1.6  $g(w_n) = (h \circ f \circ h^{-1})(w_n)$

We make the following definition.

Here  $z_n = f^n(z_0)$ , where  $f(z) = z^2 + 2z$ .

Thus  $w_n = h(z_n)$ , where  $w = h(z) = z + 1$ .

Here  $w_{n+1} = g(w_n)$ , where  $g(w) = w^2$ .

Example 1.3(b)

The inverse function  $h^{-1}$  exists because  $h$  is one-one.

**Definition** The functions  $f$  and  $g$  are **conjugate** to each other if

$$g = h \circ f \circ h^{-1},$$

for some one-one function  $h$  called the **conjugating function**. If the sequence  $\{z_n\}$  is defined by

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \dots,$$

for some  $z_0$ , and  $w_n = h(z_n)$ , for  $n = 0, 1, 2, \dots$ , then the sequence  $\{w_n\}$  satisfies

$$w_{n+1} = g(w_n), \quad \text{for } n = 0, 1, 2, \dots,$$

and  $\{z_n\}$  and  $\{w_n\}$  are called **conjugate iteration sequences**.

Do not confuse this use of the word 'conjugate', which is borrowed from group theory, with 'complex conjugate'.

**Remark** In practice, the function  $g$  is usually found by substituting  $z_n = h^{-1}(w_n)$  and  $z_{n+1} = h^{-1}(w_{n+1})$  into  $z_{n+1} = f(z_n)$  and then rearranging, as we did with (1.3).

Since the sequence  $\{w_n\}$  is the image of the sequence  $\{z_n\}$  under the function  $h$ , it follows that both sequences have the same behaviour. For example, if both  $h$  and  $h^{-1}$  are continuous functions, then  $\{z_n\}$  is convergent if and only if  $\{w_n\}$  is convergent.

Substituting  $w_n = h(z_n)$ , for  $n = 0, 1, 2, \dots$ , where  $h$  is a one-one function, amounts to making a change of variable.

Also,  $\alpha$  is a fixed point of  $f$  if and only if  $h(\alpha)$  is a fixed point of  $g$ .

If the conjugating function  $h$  is required to be one-one and entire, then it must in fact be of the form  $h(z) = az + b$ , where  $a \neq 0$ . (This can be proved using the Casorati-Weierstrass Theorem.) When dealing with the iteration of rational functions (as in the next subsection), we also use extended Möbius transformations as conjugating functions, since these are one-one on  $\widehat{\mathbb{C}}$ .

### Problem 1.8

Prove that the iteration sequence

$$z_{n+1} = z_n - z_n^2, \quad n = 0, 1, 2, \dots,$$

is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + \frac{1}{4}, \quad n = 0, 1, 2, \dots,$$

with conjugating function  $h(z) = -z + \frac{1}{2}$ . If  $z_0 = \frac{1}{2}$ , what is  $w_0$ ?

### Problem 1.9

(a) Prove that the iteration sequence

$$z_{n+1} = az_n + b, \quad n = 0, 1, 2, \dots,$$

where  $a \neq 1$ , is conjugate to the iteration sequence

$$w_{n+1} = aw_n, \quad n = 0, 1, 2, \dots,$$

with conjugating function  $h(z) = z + b/(a - 1)$ .

(b) Hence obtain a formula for  $z_n$  in terms of  $z_0$  and describe the behaviour of the sequence  $\{z_n\}$  when

- (i)  $|a| < 1$ ;      (ii)  $|a| = 1, a \neq 1$ ;      (iii)  $|a| > 1$ .

You dealt with the case  $a = 1$  in Problem 1.3(a).



## 1.4 The Newton–Raphson method

We shall now see how the ideas introduced so far help with the Newton–Raphson method, described in the Introduction. If  $p$  is a polynomial function, then the corresponding Newton–Raphson iteration sequence is

$$z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)}, \quad n = 0, 1, 2, \dots$$

We call the function to be iterated here the **Newton–Raphson function** corresponding to  $p$ , and denote it by  $N$ ; thus

$$N(z) = z - \frac{p(z)}{p'(z)}.$$

In general,  $N$  is a rational function (unless  $p$  is of degree 1) and has extension  $\hat{N}$  to  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . Thus a Newton–Raphson iteration sequence may include the point at infinity among its terms.

A key observation is that if  $\alpha$  is a simple zero of  $p$ , then  $p'(\alpha) \neq 0$  and  $N(\alpha) = \alpha$ . Thus  $\alpha$  is a fixed point of  $N$ . To classify it, we evaluate  $N'(\alpha)$ :

$$\begin{aligned} N'(\alpha) &= 1 - \frac{(p'(\alpha))^2 - p(\alpha)p''(\alpha)}{(p'(\alpha))^2} \\ &= 1 - \frac{(p'(\alpha))^2}{(p'(\alpha))^2} \quad (\text{since } p(\alpha) = 0) \\ &= 0. \end{aligned}$$

Thus a simple zero  $\alpha$  of  $p$  is a super-attracting fixed point for the Newton–Raphson function  $N$ . This is good news because it means that the Newton–Raphson method always converges rapidly to a simple zero  $\alpha$  of  $p$  provided that our initial guess is close enough to  $\alpha$ .

In 1879, Cayley analysed the Newton–Raphson method when  $p$  is the quadratic function

$$p(z) = z^2 + az + b, \quad \text{where } a, b \in \mathbb{C}.$$

If the function  $p$  has distinct zeros at  $\alpha$  and  $\beta$ , say, then these zeros must be simple. Thus  $\alpha$  and  $\beta$  are super-attracting fixed points of the Newton–Raphson function  $N$ . By Theorem 1.1, there are open discs around  $\alpha$  and  $\beta$  whose points are attracted to  $\alpha$  and  $\beta$ , respectively, under iteration of  $N$ . Cayley wished to know which points in  $\mathbb{C}$  are attracted to  $\alpha$  under iteration of  $N$ , and which are attracted to  $\beta$ . In other words, what are the basins of attraction of  $\alpha$  and  $\beta$  under  $N$ ?

Cayley found that the answer is remarkably simple: the perpendicular bisector of the line segment joining  $\alpha$  and  $\beta$  forms a ‘watershed’ for this iteration process (see Figure 1.7). If  $z_0$  falls on the same side of the watershed as  $\alpha$ , then  $N^n(z_0) \rightarrow \alpha$  as  $n \rightarrow \infty$ , but if  $z_0$  falls on the other side, then  $N^n(z_0) \rightarrow \beta$  as  $n \rightarrow \infty$ . If  $z_0$  falls exactly on the watershed line, then the sequence  $\{N^n(z_0)\}$  remains on the line!

If we consider the Newton–Raphson function for  $p(z) = z^2 + az + b$ ,

$$\begin{aligned} N(z) &= z - \frac{p(z)}{p'(z)} \\ &= z - \frac{z^2 + az + b}{2z + a} \\ &= \frac{z^2 - b}{2z + a}, \end{aligned}$$

then it is not at all evident why Cayley’s result should be true. However, we obtain a much simpler iteration sequence by using the conjugating function

$$h(z) = \frac{z - \alpha}{z - \beta},$$

which is a Möbius transformation. The extended Möbius transformation  $\hat{h}$  is one-one on  $\hat{\mathbb{C}}$  and maps  $\alpha$  to 0 and  $\beta$  to  $\infty$ . Also  $h$  has the remarkable property

Unit D1, Section 1

Unit B3, Section 5

If  $\alpha$  is a zero of order  $k$ , where  $k > 1$ , then it can be shown that  $\alpha$  is a fixed point of  $N$  such that

$$N'(\alpha) = 1 - 1/k < 1,$$

so the Newton–Raphson method still works, but not so well. See, for example, Problem 1.11.

Arthur Cayley (1821–1895) is well known for pioneering work in group theory and linear algebra.

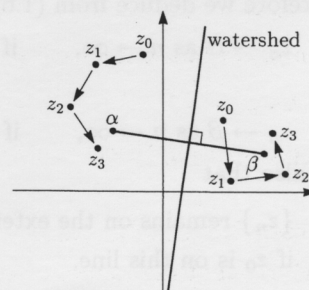


Figure 1.7

Unit D1, Section 2

that

$$h(N(z)) = (h(z))^2,$$

which we ask you to prove in Problem 1.10.

Thus, if

$$z_{n+1} = N(z_n), \quad n = 0, 1, 2, \dots,$$

and if

$$w_n = h(z_n), \quad \text{for } n = 0, 1, 2, \dots,$$

then

$$\begin{aligned} w_{n+1} &= h(z_{n+1}) \\ &= h(N(z_n)) \\ &= (h(z_n))^2 \quad (\text{by Equation (1.5)}) \\ &= w_n^2, \quad \text{for } n = 0, 1, 2, \dots \end{aligned}$$

Thus

$$w_n = g^n(w_0), \quad \text{for } n = 1, 2, \dots,$$

where  $g(w) = w^2$ .

Now we know that

$$w_n \rightarrow 0 \text{ as } n \rightarrow \infty, \quad \text{if } |w_0| < 1,$$

that

$$w_n \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \text{if } |w_0| > 1,$$

and also that

the sequence  $\{w_n\}$  remains on the circle  $\{w : |w| = 1\}$  if  $|w_0| = 1$ .

To find out what happens to the original sequence  $\{z_n\}$ , we note that

$$h^{-1}(w_n) = z_n, \quad \hat{h}^{-1}(0) = \alpha, \quad \hat{h}^{-1}(\infty) = \beta$$

and

$$|w_0| = |h(z_0)| = \left| \frac{z_0 - \alpha}{z_0 - \beta} \right|.$$

Therefore we deduce from (1.6) and (1.7) that

$$z_n \rightarrow \alpha \text{ as } n \rightarrow \infty, \quad \text{if } |z_0 - \alpha| < |z_0 - \beta|,$$

that

$$z_n \rightarrow \beta \text{ as } n \rightarrow \infty, \quad \text{if } |z_0 - \alpha| > |z_0 - \beta|,$$

and also that

$\{z_n\}$  remains on the extended line  $\{z : |z - \alpha| = |z - \beta|\} \cup \{\infty\}$   
if  $z_0$  is on this line.

This is Cayley's remarkable solution.

### Problem 1.10

Prove Identity (1.5).

(Hint: Note that both  $z = \alpha$  and  $z = \beta$  satisfy the equation  $z^2 = -(az + b)$ .)

### Problem 1.11

Describe what happens under iteration of  $N$  if  $\alpha = \beta$ .

Finally, we look briefly at the Newton–Raphson method for cubic polynomial functions. Here, you might guess that the complex plane divides itself into three simple regions, each surrounding one zero of  $p$  and consisting of those points which are attracted to that zero under iteration of the Newton–Raphson function  $N$ . Such indeed seems to have been Cayley's hunch in 1879, although he was unable to prove such a result. With the help of computer-generated

(1.5) Identity (1.5) may be written in the form

$$N = h^{-1} \circ g \circ h, \quad \text{where } g(w) = w^2.$$

(1.6) Example 1.3(b)  
and  
Problem 1.4(d)

(1.7)

Note that the function

$$h^{-1}(w) = \frac{-\beta w + \alpha}{-w + 1}$$

is analytic at 0 and has a removable singularity at  $\infty$ .



pictures we can now see why Cayley had no chance of finding a simple solution to this problem!

Consider  $p(z) = z^3 - 1$ , whose zeros are

$$\alpha_1 = 1, \quad \alpha_2 = e^{2\pi i/3} = \frac{1}{2}(-1 + i\sqrt{3}), \quad \alpha_3 = e^{4\pi i/3} = \frac{1}{2}(-1 - i\sqrt{3}).$$

In this case

$$\begin{aligned} N(z) &= z - \frac{z^3 - 1}{3z^2} \\ &= \frac{2z^3 + 1}{3z^2} \end{aligned}$$

and, as before,  $\alpha_1, \alpha_2, \alpha_3$  are each super-attracting fixed points of  $N$ .

Figure 1.8 shows in white the basin of attraction of the fixed point  $\alpha_1 = 1$ .

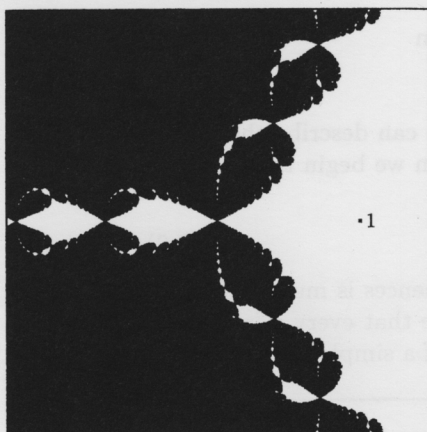


Figure 1.8 The basin of attraction of 1

The basins of attraction of  $\alpha_1, \alpha_2, \alpha_3$  must be congruent to each other under rotation about 0 through  $2\pi/3$ , because of the symmetry of  $\alpha_1, \alpha_2, \alpha_3$ . But these basins are not at all simple, and they are not even regions (because they are, in fact, not connected). The union of these three strange basins is almost the whole of  $\mathbb{C}$ . In addition, there is a complicated 'watershed' which separates the basins of attraction (see Figure 1.9) and which manages, somehow, to be the boundary of all three sets simultaneously!

Thus the iteration of even fairly simple rational functions can lead to very complicated behaviour. In the next section, we find that complicated behaviour can occur even for the iteration of simple polynomial functions.

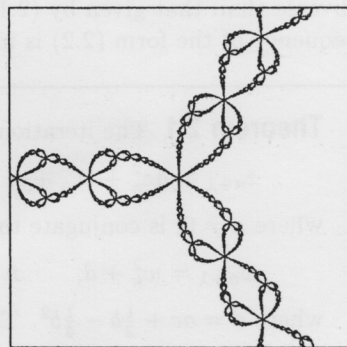


Figure 1.9 The watershed

We return to the Newton-Raphson method in Section 5.

## 2 ITERATING COMPLEX QUADRATICS



After working through this section, you should be able to:

- conjugate a given quadratic iteration sequence to one determined by a member of the family of functions  $\{P_c : c \in \mathbb{C}\}$ , where  $P_c(z) = z^2 + c$ ;
- explain why  $P_c^n(z_0) \rightarrow \infty$  as  $n \rightarrow \infty$  if  $|z_0|$  is large enough;
- describe some of the properties of the *escape set*  $E_c$  of  $P_c$  and of its complement, the *keep set*  $K_c$ ;
- find the *periodic points* of  $P_c$ , for certain values of  $c$ , and find their nature;
- understand the definition of the *Julia set*  $J_c$ .

### 2.1 The basic quadratic family

In Section 1 we saw that iteration sequences of the form

$$z_{n+1} = az_n + b, \quad n = 0, 1, 2, \dots, \quad (2.1)$$

can be completely analysed; that is, for all  $a, b \in \mathbb{C}$ , we can describe the behaviour of  $\{z_n\}$  for any initial term  $z_0$ . In this section we begin to study iteration sequences of the form

$$z_{n+1} = az_n^2 + bz_n + c, \quad n = 0, 1, 2, \dots, \quad (2.2)$$

where  $a \neq 0$ . We shall find that the family of such sequences is much more diverse than that given by (2.1). To begin with we note that every iteration sequence of the form (2.2) is in fact conjugate to one of a simpler type.

Example 1.3(c) and Problem 1.9

**Theorem 2.1** The iteration sequence

$$z_{n+1} = az_n^2 + bz_n + c, \quad n = 0, 1, 2, \dots,$$

where  $a \neq 0$ , is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \dots, \quad (2.3)$$

where  $d = ac + \frac{1}{2}b - \frac{1}{4}b^2$ . The conjugating function is

$$h(z) = az + \frac{1}{2}b.$$

Problem 1.8 is a special case of this result, with  $a = -1$ ,  $b = 1$ ,  $c = 0$ .

**Proof** The recurrence relation (2.2) can be rearranged as

$$az_{n+1} = (az_n + \frac{1}{2}b)^2 + ac - \frac{1}{4}b^2, \quad n = 0, 1, 2, \dots.$$

Thus, putting  $w_n = h(z_n)$ , for  $n = 0, 1, 2, \dots$ , where  $h(z) = az + \frac{1}{2}b$ , we obtain

$$w_{n+1} - \frac{1}{2}b = w_n^2 + ac - \frac{1}{4}b^2, \quad n = 0, 1, 2, \dots;$$

that is,

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \dots,$$

where  $d = ac + \frac{1}{2}b - \frac{1}{4}b^2$ , as required. ■

Multiply by  $a$  and complete the square.

There are many different iteration sequences of the form (2.2) which are conjugate to any one iteration sequence of the form (2.3). This is illustrated in the following problem.

#### Problem 2.1

Use Theorem 2.1 to show that each of the following iteration sequences

- $z_{n+1} = 4z_n(1 - z_n)$ ,  $n = 0, 1, 2, \dots$ , with  $z_0 = \frac{1}{2}$ ,
- $z_{n+1} = 1 - 2z_n^2$ ,  $n = 0, 1, 2, \dots$ , with  $z_0 = 0$ ,

is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 - 2, \quad n = 0, 1, 2, \dots, \text{ with } w_0 = 0.$$



Theorem 2.1 tells us that if we wish to understand the possible behaviour of quadratic functions under iteration, then it is sufficient to consider only those of the form (2.3), and it is convenient to relabel these as

$$z_{n+1} = z_n^2 + c, \quad n = 0, 1, 2, \dots,$$

where  $c$  is a complex parameter. We shall devote most of the rest of this unit to such iteration sequences, and so we introduce a name for the corresponding quadratic functions.

**Definition** The set of functions  $\{P_c : c \in \mathbb{C}\}$  defined by

$$P_c(z) = z^2 + c,$$

where  $c \in \mathbb{C}$ , is the **family of basic quadratic functions**.

For example,  $P_0(z) = z^2$ ,  $P_1(z) = z^2 + 1$  and  $P_i(z) = z^2 + i$  are all basic quadratic functions. In the following problems we ask you to establish some elementary properties of the basic quadratic functions.

### Problem 2.2

- Determine the rules for the functions  $P_c^2$  and  $P_c^3$ .
- Write down a formula for  $P_c^{n+1}(z)$  in terms of  $P_c^n(z)$ , and hence prove that  $P_c^n$  is an even polynomial function of degree  $2^n$ .

Remember that  $P_c^n$  denotes the  $n$ th iterate of  $P_c$ .

### Problem 2.3

- Show that the fixed points of  $P_c$  are  $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$ , and prove that at least one of these is repelling (unless  $c = \frac{1}{4}$ ).  
(Hint: If  $\alpha$  and  $\beta$  are the fixed points of  $P_c$ , then  $\frac{1}{2}(P'_c(\alpha) + P'_c(\beta)) = 1$ .)
- What happens if  $c = \frac{1}{4}$ ?

## 2.2 The escape set and the keep set

In Example 1.2(b) we observed that the iteration sequence

$$z_n = P_0^n(z_0), \quad n = 0, 1, 2, \dots,$$

can be determined explicitly as

$$z_n = z_0^{2^n}, \quad \text{for } n = 0, 1, 2, \dots$$

Thus

$$\left. \begin{aligned} z_n &\rightarrow 0 \text{ as } n \rightarrow \infty, & \text{if } |z_0| < 1, \\ z_n &\rightarrow \infty \text{ as } n \rightarrow \infty, & \text{if } |z_0| > 1, \\ |z_n| &= 1, \text{ for } n = 1, 2, \dots, & \text{if } |z_0| = 1. \end{aligned} \right\} \quad (2.4)$$

It is natural to ask whether a similar behaviour occurs for other values of  $c$ . We shall see later that when the initial term  $z_0$  is small, the sequence  $z_n = P_c^n(z_0)$ ,  $n = 1, 2, \dots$ , behaves in dramatically different ways for different values of  $c$ . However, when the initial term  $z_0$  is large these sequences behave in essentially the same way, for all values of  $c$ , as we now show.

Here

$$P_0(z) = z^2, \quad \text{so that } c = 0.$$

**Theorem 2.2** Let  $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$ . Then, for  $|z_0| > r_c$ ,

$\{|P_c^n(z_0)|\}$  is an increasing sequence,

and

$$P_c^n(z_0) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

If  $c = 0$ , then  $r_c = 1$  and we recover the middle line of (2.4).

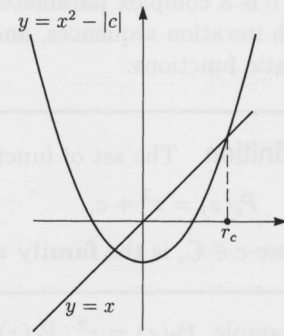


Figure 2.1  $r_c^2 - |c| = r_c$

**Proof** First note that, by the backwards form of the Triangle Inequality,

$$|P_c(z)| = |z^2 + c| \geq |z|^2 - |c|. \quad (2.5)$$

The number  $r_c$  is the positive solution of the equation  $x^2 - |c| = x$  (see Figure 2.1), and we claim that if  $\varepsilon > 0$ , then

$$x^2 - |c| \geq (1 + \varepsilon)x, \quad \text{for } x \geq r_c + \varepsilon. \quad (2.6)$$

Indeed, if  $x \geq r_c + \varepsilon$ , then

$$\begin{aligned} \frac{x^2 - |c|}{x} &= x - \frac{|c|}{x} \\ &\geq (r_c + \varepsilon) - \frac{|c|}{r_c + \varepsilon} \\ &= \frac{r_c^2 + 2r_c\varepsilon + \varepsilon^2 - |c|}{r_c + \varepsilon} \\ &= \frac{r_c + 2r_c\varepsilon + \varepsilon^2}{r_c + \varepsilon} \quad (\text{since } r_c^2 - |c| = r_c) \\ &\geq \frac{r_c + r_c\varepsilon + \varepsilon + \varepsilon^2}{r_c + \varepsilon} \quad (\text{since } r_c \geq 1) \\ &= 1 + \varepsilon, \end{aligned}$$

as required for Inequality (2.6). Inequalities (2.5) and (2.6) now give

$$|P_c(z)| \geq (1 + \varepsilon)|z| > |z|, \quad \text{for } |z| \geq r_c + \varepsilon.$$

If  $|z_0| \geq r_c + \varepsilon$ , then we can apply this inequality successively to  $z_0, P_c(z_0), P_c^2(z_0), \dots$ , to deduce that the sequence  $\{|P_c^n(z_0)|\}$  is increasing, and

$$|P_c^n(z_0)| \geq (1 + \varepsilon)^n |z_0|, \quad \text{for } n = 1, 2, \dots$$

Hence  $P_c^n(z_0) \rightarrow \infty$  as  $n \rightarrow \infty$ . Since  $\varepsilon$  is any positive number, the proof is complete. ■

## Remarks

1 Theorem 2.2 may be easier to understand if you think of  $\infty$  as a fixed point of the extended function  $\hat{P}_c$  of  $P_c$ . To discover the nature of this fixed point, we use the conjugating function  $h(z) = 1/z$ , which moves the fixed point from  $\infty$  to 0. By this means the iteration sequence

$$z_{n+1} = z_n^2 + c, \quad n = 0, 1, 2, \dots,$$

is conjugate to the iteration sequence  $\{w_n\}$ , where

$$\frac{1}{w_{n+1}} = \left(\frac{1}{w_n}\right)^2 + c, \quad n = 0, 1, 2, \dots;$$

that is,

$$w_{n+1} = \frac{w_n^2}{1 + cw_n^2}, \quad n = 0, 1, 2, \dots$$

As expected, because  $\hat{P}_c(\infty) = \infty$ , the corresponding function  $Q_c(w) = w^2/(1 + cw^2)$  has a fixed point at 0, which turns out to be super-attracting. Thus, by Theorem 1.1,

$$w_n = Q_c^n(w_0) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ provided that } |w_0| \text{ is small enough,}$$

and so

$$z_n = P_c^n(z_0) \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ provided that } |z_0| \text{ is large enough.}$$

$$\begin{aligned} w_n &= h(z_n) = \frac{1}{z_n} \\ \implies z_n &= \frac{1}{w_n} \end{aligned}$$

Since  $Q'_c(w) = \frac{2w}{(1 + cw^2)^2}$ , we have  $Q'_c(0) = 0$ .



2 The formula  $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$  will play a significant role later in the unit; therefore we indicate in Figure 2.2 the graph of the function  $|c| \mapsto r_c$ , together with the graph of the identity function  $|c| \mapsto |c|$  for comparison. Notice, in particular, that

$$|c| \leq r_c \iff |c| \leq 2 \quad (2.7)$$

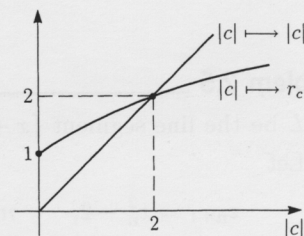


Figure 2.2

### Problem 2.4

(a) Verify that

$$r_0 = 1, \quad r_i = \frac{1}{2}(1 + \sqrt{5}) \quad \text{and} \quad r_{-2} = 2.$$

(b) Show that the sequences  $\{P_0^n(1)\}$  and  $\{P_{-2}^n(2)\}$  are constant. What does this tell you about the values  $r_0$  and  $r_{-2}$  in relation to Theorem 2.2?

We now investigate the set of *all* points which are attracted to  $\infty$  or ‘escape’ to  $\infty$  under iteration of  $P_c$ . We call this set the *escape set*; its complement is the set of points which we ‘keep’ under iteration of  $P_c$ .

**Definition** For  $c \in \mathbb{C}$ , the **escape set**  $E_c$  is

$$E_c = \{z : P_c^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

The **keep set**  $K_c$  is the complement of  $E_c$ .

**Remark** The escape set  $E_c$  may be thought of as the basin of attraction of  $\infty$ .

For example, we know by (2.4) that

$$\begin{aligned} E_0 &= \{z : P_0^n(z) = z^{2^n} \rightarrow \infty \text{ as } n \rightarrow \infty\} \\ &= \{z : |z| > 1\}, \end{aligned}$$

and hence that

$$\begin{aligned} K_0 &= \mathbb{C} - E_0 \\ &= \{z : |z| \leq 1\} \end{aligned}$$

(see Figure 2.3).

This example is rather misleading as, for most values of  $c$ ,  $E_c$  does not have a simple shape. Figure 2.4 shows several examples of sets  $E_c$  (in white!) and  $K_c$ , plotted by computer.

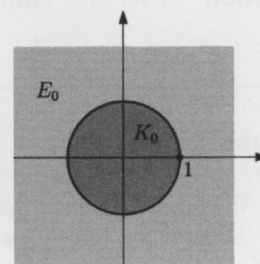
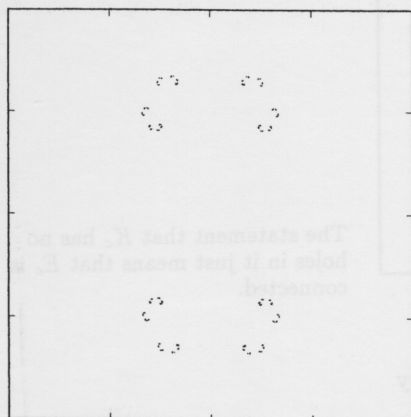


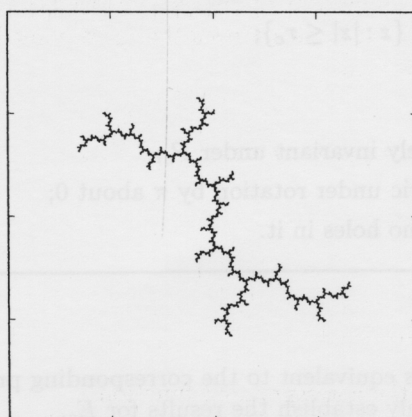
Figure 2.3

Here, and throughout this section, the square represented in such figures is

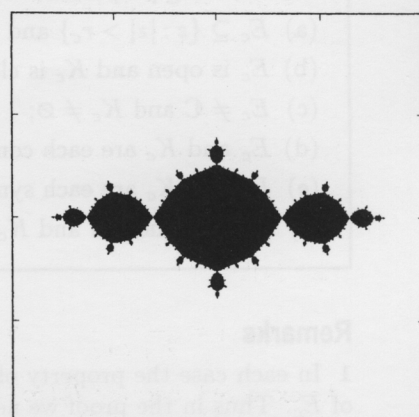
$$\{z : -2 \leq \operatorname{Re} z, \operatorname{Im} z \leq 2\}.$$



(a)  $E_c$  and  $K_c : c = 1$



(b)  $E_c$  and  $K_c : c = i$



(c)  $E_c$  and  $K_c : c = -1$

Figure 2.4

In fact,  $c = -2$  is the only other value for which  $E_c$  and  $K_c$  have a simple shape. We now ask you to investigate this case.

## Problem 2.5

Let  $L$  be the line segment  $\{x + iy : |x| \leq 2, y = 0\}$ .

$L$  is the interval  $[-2, 2]$ .

(a) Let

$$z_{n+1} = z_n^2 - 2, \quad n = 0, 1, 2, \dots$$

Prove that

if  $z_0 \in L$ , then  $z_n \in L$ , for  $n = 1, 2, \dots$ ;

if  $z_0 \in \mathbb{C} - L$ , then  $z_n \in \mathbb{C} - L$ , for  $n = 1, 2, \dots$ .

(b) Prove that if  $z_0 \in \mathbb{C} - L$ , then the sequence  $\{z_n\}$  in part (a) is conjugate to an iteration sequence of the form

$$w_{n+1} = w_n^2, \quad n = 0, 1, 2, \dots,$$

and deduce that  $z_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

(Hint: Recall from Unit D1, Subsection 4.4, that the Joukowski function  $J(w) = w + 1/w$  maps  $\{w : |w| > 1\}$  one-one and conformally onto  $\mathbb{C} - L$ , put  $w_n = J^{-1}(z_n)$ , for  $n = 0, 1, 2, \dots$ , and verify that  $J(w_{n+1}) = J(w_n^2)$ .)

(c) Deduce that  $E_{-2} = \mathbb{C} - L$  and  $K_{-2} = L$  (see Figure 2.5).

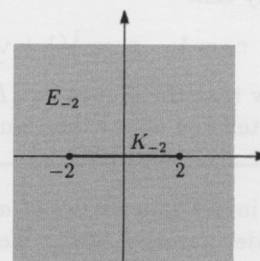


Figure 2.5

Though the set  $E_c$  is usually very complicated, a number of general observations can be made about it. For example, we can show that  $E_c$  has the property of being *completely invariant* under  $P_c$ .

**Definition** A set  $A$  is **completely invariant** under a function  $f$  if

$$z \in A \iff f(z) \in A.$$

This means that if  $z$  lies in  $A$ , then  $f(z)$  and all its iterates also lie in  $A$  and, moreover, any point whose iterates eventually lie in  $A$  must itself lie in  $A$ . For example, it is easy to check that each of the sets  $\mathbb{C}$ ,  $\{0\}$ ,  $\mathbb{C} - \{0\}$ ,  $\{z : |z| < 1\}$ ,  $\{z : |z| = 1\}$  and  $\{z : |z| > 1\}$  is completely invariant under the function  $f(z) = z^2$ .

The following result lists several key facts about  $E_c$  and  $K_c$ .

**Theorem 2.3** For each  $c \in \mathbb{C}$ , the escape set  $E_c$  and the keep set  $K_c$  have the following properties:

- (a)  $E_c \supseteq \{z : |z| > r_c\}$  and  $K_c \subseteq \{z : |z| \leq r_c\}$ ;
- (b)  $E_c$  is open and  $K_c$  is closed;
- (c)  $E_c \neq \mathbb{C}$  and  $K_c \neq \emptyset$ ;
- (d)  $E_c$  and  $K_c$  are each completely invariant under  $P_c$ ;
- (e)  $E_c$  and  $K_c$  are each symmetric under rotation by  $\pi$  about 0;
- (f)  $E_c$  is connected and  $K_c$  has no holes in it.

The statement that  $K_c$  has no holes in it just means that  $E_c$  is connected.

## Remarks

1 In each case the property of  $K_c$  is equivalent to the corresponding property of  $E_c$ . Thus in the proof we need only establish the results for  $E_c$ .

2 Note that Properties (a), (b) and (c) combine to tell us that  $K_c$  is a non-empty compact set.



### Proof of Theorem 2.3

- (a) We have already proved part (a) in Theorem 2.2.  
 (b) Suppose that  $z_0 \in E_c$ . Then, by definition,  $P_c^n(z_0) \rightarrow \infty$  as  $n \rightarrow \infty$ . Thus, for some  $n_0$ ,  $|P_c^{n_0}(z_0)| > r_c$ .

Let  $\varepsilon = |P_c^{n_0}(z_0)| - r_c$ . Since  $\varepsilon > 0$  and  $P_c^{n_0}$  is a polynomial function,  $P_c^{n_0}$  is continuous at  $z_0$  and so there exists  $\delta > 0$  such that

$$|z - z_0| < \delta \implies |P_c^{n_0}(z) - P_c^{n_0}(z_0)| < \varepsilon$$

and hence

$$|z - z_0| < \delta \implies |P_c^{n_0}(z)| > r_c$$

(see Figure 2.6).

It follows from Theorem 2.2 that

$$|z - z_0| < \delta \implies P_c^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty,$$

so that  $\{z : |z - z_0| < \delta\} \subseteq E_c$ . This shows that  $E_c$  is open.

- (c) The set  $E_c$  is not the whole of  $\mathbb{C}$ , because it does not contain the fixed points of  $P_c$ , found in Problem 2.3. (These must lie in  $K_c$ .)  
 (d) We ask you to prove part (d) in Problem 2.6.  
 (e) Since

$$P_c^n(-z) = P_c^n(z), \quad \text{for } n = 1, 2, \dots,$$

because  $P_c^n$  is an even function (see Problem 2.2(b)), we have

$$P_c^n(-z) \rightarrow \infty \iff P_c^n(z) \rightarrow \infty$$

and so

$$-z \in E_c \iff z \in E_c.$$

- (f) Finally, we show that  $E_c$  is connected, that is, each pair of points in  $E_c$  can be joined by a path lying in  $E_c$ . Since the set  $A_c = \{z : |z| > r_c\}$  is connected and  $A_c \subseteq E_c$ , it is sufficient to show that each point  $\alpha$  in  $E_c$  can be joined to some point of  $A_c$  by a path in  $E_c$ . We prove this by contradiction.

Suppose in fact that  $\alpha$  is a point of  $E_c$  which *cannot* be joined to the set  $A_c$  in this way. We define the following set:

$$\mathcal{R} = \{z \in E_c : z \text{ can be joined to } \alpha \text{ by a path in } E_c\}$$

(see Figure 2.7).

Then  $\mathcal{R} \neq \emptyset$  (since  $\alpha \in \mathcal{R}$ ),  $\mathcal{R}$  is open (because if  $z$  can be joined to  $\alpha$  in  $E_c$ , then so can points of any open disc in  $E_c$  with centre  $z$ ) and  $\mathcal{R}$  is connected (because pairs of points in  $\mathcal{R}$  can be joined in  $\mathcal{R}$  via  $\alpha$ ). Thus  $\mathcal{R}$  is a subregion of  $E_c$ . Since  $\alpha$  cannot be joined in  $\mathcal{R}$  to  $A_c$  and  $\mathcal{R}$  is open, we deduce that  $\mathcal{R} \subseteq \{z : |z| < r_c\}$ .

We can now use the Maximum Principle. If  $\beta \in \partial\mathcal{R}$ , then  $\beta$  does not lie in  $E_c$  (since, otherwise, we could enlarge  $\mathcal{R}$  slightly). Thus

$$|P_c^n(\beta)| \leq r_c, \quad \text{for } n = 1, 2, \dots$$

By applying the Maximum Principle to each of the polynomial functions  $P_c^n$  on  $\mathcal{R}$ , we obtain

$$|P_c^n(z)| \leq r_c, \quad \text{for } n = 1, 2, \dots, \text{ and } z \in \mathcal{R},$$

which contradicts the fact that  $\mathcal{R} \subseteq E_c$ . Hence  $E_c$  is connected. ■

This proof may be omitted on a first reading.

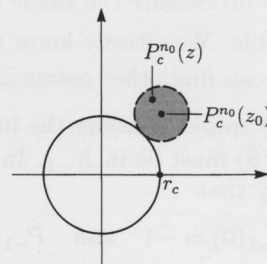


Figure 2.6

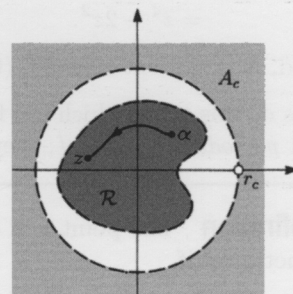


Figure 2.7

### Problem 2.6

Prove part (d) of Theorem 2.3.

Unit C2, Theorem 4.2

## 2.3 Periodic points

Figure 2.4 shows that the keep sets  $K_c$  have a remarkably diverse form. In order to investigate the shape of  $K_c$ , we need to identify as many points in  $K_c$  as possible. We already know that the fixed points  $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$  of  $P_c$  lie in  $K_c$ , and we can find other points of  $K_c$  by generalizing the notion of a fixed point.

As an example, consider the function  $P_{-1}(z) = z^2 - 1$ , whose two fixed points  $\frac{1}{2}(1 \pm \sqrt{5})$  must lie in  $K_{-1}$ . In addition, the points 0 and  $-1$ , have the property that

$$P_{-1}(0) = -1 \quad \text{and} \quad P_{-1}(-1) = 0 \quad (2.8)$$

(see Figure 2.8). Thus the sequence  $P_{-1}^n(0), n = 0, 1, 2, \dots$ , cycles endlessly between the points 0 and  $-1$ , as does the sequence  $P_{-1}^n(-1), n = 0, 1, 2, \dots$ . This means that 0 and  $-1$  must both lie in  $K_{-1}$ . Since  $K_{-1}$  is symmetric under a rotation by  $\pi$  about 0, we can begin to build up a picture of  $K_{-1}$  (see Figure 2.9). This is very far from the complicated set in Figure 2.4(c), but at least it is a start!

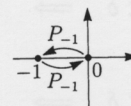


Figure 2.8

Theorem 2.3(e)

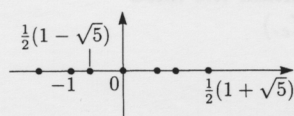


Figure 2.9

The fact that 0 and  $-1$  satisfy Equations (2.8) can be interpreted as saying that 0 and  $-1$  are both fixed points of the second iterate

$$\begin{aligned} P_{-1}^2(z) &= P_{-1}(z^2 - 1) \\ &= (z^2 - 1)^2 - 1 \\ &= z^4 - 2z^2. \end{aligned}$$

Indeed, it is evident that  $P_{-1}^2(0) = 0$  and  $P_{-1}^2(-1) = -1$ .

Points of this type, which are fixed points of higher iterates of a function  $f$ , are called *periodic points* of  $f$ ; they repeat periodically under iteration of  $f$ .

**Definition** The point  $\alpha$  is a **periodic point**, with period  $p$ , of a function  $f$  if

$$f^p(\alpha) = \alpha, \text{ but } f^k(\alpha) \neq \alpha, \text{ for } k = 1, 2, \dots, p-1.$$

The  $p$  points

$$\alpha, f(\alpha), f^2(\alpha), \dots, f^{p-1}(\alpha)$$

then form a **cycle of period  $p$** , or a  **$p$ -cycle** of  $f$  (see Figure 2.10).

Thus  $p$  is the smallest positive integer such that

$$f^p(\alpha) = \alpha.$$

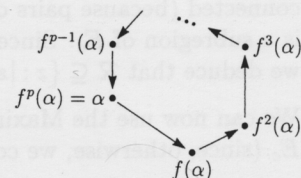


Figure 2.10

### Remarks

1 Note that if we apply  $f$  repeatedly to points of a  $p$ -cycle, then we just obtain points of the  $p$ -cycle. Also the points of a  $p$ -cycle must be distinct. Indeed, if we had

$$f^k(\alpha) = f^\ell(\alpha), \quad \text{where } 0 \leq k < \ell \leq p-1,$$

then the point  $f^p(\alpha) = f^{p-k}(f^k(\alpha))$  would have to lie among the terms

$$f^{k+1}(\alpha), \dots, f^\ell(\alpha),$$

which do not include  $\alpha$ , which is a contradiction. Therefore, the points of a  $p$ -cycle are each distinct periodic points of  $f$  with period  $p$ .

2 Note that a fixed point of  $f$  is a 1-cycle of  $f$ .

3 Evidently, all the periodic points of  $P_c$  lie in the keep set  $K_c$ .



Determining the periodic points with period  $p$  of a given function  $f$ , for a given  $p > 1$ , is usually more difficult than determining the fixed points of  $f$ . This is because the equation

$$f^p(z) = z \quad (2.9)$$

usually has many more solutions as  $p$  increases and the rule for  $f^p$  is usually more complicated. Notice, however, that not all solutions of Equation (2.9) need be periodic points with period  $p$ . For example, any fixed point of  $f$  also satisfies Equation (2.9). More generally, if  $q$  is a factor of  $p$ , then any solution of  $f^q(z) = z$  is also a solution of  $f^p(z) = z$ .

### Example 2.1

Determine all periodic points with period 2 of the function  $P_0(z) = z^2$ , and write down the corresponding 2-cycles.

#### Solution

Since  $P_0^2(z) = (z^2)^2 = z^4$ , we have to solve the equation  $z^4 = z$ :

$$\begin{aligned} z^4 = z &\iff z^4 - z = 0 \\ &\iff z(z^3 - 1) = 0. \end{aligned}$$

The solutions of this quartic equation are  $0, 1, e^{2\pi i/3} = \frac{1}{2}(-1 + \sqrt{3}i)$  and  $e^{4\pi i/3} = \frac{1}{2}(-1 - \sqrt{3}i)$ . Of these, the points  $0, 1$  are fixed points of  $P_0$ , whereas

$$P_0(e^{2\pi i/3}) = (e^{2\pi i/3})^2 = e^{4\pi i/3},$$

and

$$P_0(e^{4\pi i/3}) = (e^{4\pi i/3})^2 = e^{2\pi i/3}.$$

Hence both  $e^{2\pi i/3}$  and  $e^{4\pi i/3}$  are periodic points of  $P_0$  with period 2, and they belong to the 2-cycle  $e^{2\pi i/3}, e^{4\pi i/3}$ . ■

Note that, as expected, the 2-cycle found in this example lies in the keep set  $K_0 = \{z : |z| \leq 1\}$  (see Figure 2.11).

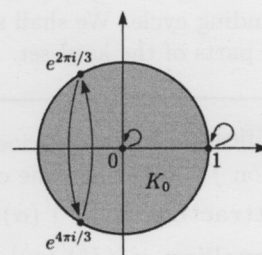


Figure 2.11

### Problem 2.7

- Prove that  $-i$  is a periodic point with period 2 of the function  $P_i(z) = z^2 + i$ . Hence find five points in  $K_i$ , none of them fixed points of  $P_i$ , and plot them.
- Determine all periodic points with period 3 of the function  $P_0(z) = z^2$ , and write down the corresponding 3-cycles. Plot all the fixed points, 2-cycles and 3-cycles of  $P_0$  on the same diagram.
- Prove that  $\frac{1}{2}(-1 + \sqrt{2})$  is a periodic point of  $P_{-5/4}$ .

The set  $K_i$  was plotted in Figure 2.4(b).

We now return to the function  $P_{-1}(z) = z^2 - 1$ , which has the 2-cycle  $0, -1$ ; that is,  $0$  and  $-1$  are fixed points of the second iterate

$$P_{-1}^2(z) = z^4 - 2z^2.$$

Since

$$\begin{aligned} (P_{-1}^2)'(z) &= 4z^3 - 4z \\ &= 4z(z^2 - 1), \end{aligned}$$

it follows that

$$(P_{-1}^2)'(0) = 0 \quad \text{and} \quad (P_{-1}^2)'(-1) = 0,$$

so both  $0$  and  $-1$  are super-attracting fixed points of  $P_{-1}^2$ . The fact that these two derivatives have the same value is no accident, as the following result shows.

**Theorem 2.4** Let  $\alpha, f(\alpha), f^2(\alpha), \dots, f^{p-1}(\alpha)$  form a  $p$ -cycle of an analytic function  $f$ . Then

$$(a) \quad (f^p)'(\alpha) = f'(\alpha) \times f'(f(\alpha)) \times f'(f^2(\alpha)) \times \dots \times f'(f^{p-1}(\alpha)) \quad (2.10)$$

and, moreover,

(b) the derivative of  $f^p$  takes the same value at each point of the  $p$ -cycle; that is,

$$(f^p)'(\alpha) = (f^p)'(f(\alpha)) = (f^p)'(f^2(\alpha)) = \dots = (f^p)'(f^{p-1}(\alpha)).$$

**Proof** Since  $f^p(z) = f(f(\dots(f(z))\dots))$ , where the function  $f$  is applied  $p$  times, we deduce from repeated applications of the Chain Rule that

$$(f^p)'(z) = f'(f^{p-1}(z)) \times \dots \times f'(f(z)) \times f'(z).$$

By putting  $z = \alpha$ , we obtain (a).

Thus  $(f^p)'(\alpha)$  is the product of the derivatives of  $f$  at the points of the  $p$ -cycle, and so  $(f^p)'(f(\alpha))$  is also the product of the derivatives of  $f$  at the points of the  $p$ -cycle, and similarly for the other points  $f^2(\alpha), \dots, f^{p-1}(\alpha)$ . This establishes (b). ■

Theorem 2.4 allows us to classify the periodic points of an analytic function  $f$ , by using the number  $(f^p)'(\alpha)$ , which is called the **multiplier** of the corresponding cycle. We shall see shortly that different types of cycle lie in different parts of the complex plane.

Unit A4, Theorem 3.1

Some texts use the name *eigenvalue* rather than multiplier.

**Definitions** If  $\alpha$  is a periodic point, with period  $p$ , of an analytic function  $f$ , then  $\alpha$  and the corresponding  $p$ -cycle are

- (a) **attracting**, if  $|(f^p)'(\alpha)| < 1$ ;
- (b) **repelling**, if  $|(f^p)'(\alpha)| > 1$ ;
- (c) **indifferent**, if  $|(f^p)'(\alpha)| = 1$ ;
- (d) **super-attracting**, if  $(f^p)'(\alpha) = 0$ .

In the next example we demonstrate two ways to calculate the multiplier.

### Example 2.2

Determine the nature of the periodic point  $e^{2\pi i/3}$  of  $P_0(z) = z^2$ .

#### Solution

In Example 2.1 we found that  $e^{2\pi i/3}$  is a periodic point, with period 2, of the function  $P_0$ . Since  $P_0^2(z) = z^4$ , the multiplier is

$$(P_0^2)'(e^{2\pi i/3}) = 4(e^{2\pi i/3})^3 \quad (\text{since } (P_0^2)'(z) = 4z^3) \\ = 4,$$

so that  $|(P_0^2)'(e^{2\pi i/3})| > 1$ . Thus  $e^{2\pi i/3}$  is a repelling periodic point of  $P_0$ .

Alternatively, the point  $e^{2\pi i/3}$  is part of the 2-cycle  $e^{2\pi i/3}, e^{4\pi i/3}$  for the function  $P_0$ . Thus, by Theorem 2.4(a), the multiplier of the 2-cycle  $e^{2\pi i/3}, e^{4\pi i/3}$  is

$$P_0'(e^{2\pi i/3}) P_0'(e^{4\pi i/3}) = (2e^{2\pi i/3})(2e^{4\pi i/3}) \quad (\text{since } P_0'(z) = 2z), \\ = 4,$$

as before. ■

Note that the second method avoids the calculation of the rule for the  $p$ th iterate, which can be involved.



## Problem 2.8

Determine the nature of each of the following periodic points, found in Problem 2.7.

- (a)  $-i$ , a periodic point of  $P_i$
- (b)  $e^{2\pi i/7}$ , a periodic point of  $P_0$
- (c)  $\frac{1}{2}(-1 + \sqrt{2})$ , a periodic point of  $P_{-5/4}$

We now look more closely at where the periodic points of  $P_c$  lie in  $K_c$ . As an example, recall that the function  $P_{-1}(z) = z^2 - 1$  has fixed points at  $\frac{1}{2}(1 \pm \sqrt{5})$ , which are both repelling because

$$\left| P'_{-1}\left(\frac{1}{2}(1 + \sqrt{5})\right) \right| = 1 + \sqrt{5} > 1 \quad \text{and} \quad \left| (P_{-1})'\left(\frac{1}{2}(1 - \sqrt{5})\right) \right| = \sqrt{5} - 1 > 1. \quad P'_{-1}(z) = 2z$$

Also, we saw before the proof of Theorem 2.4 that  $P_{-1}$  has the super-attracting 2-cycle  $0, -1$ . In Figure 2.12, these points are plotted on a computer-generated picture of  $\partial K_{-1}$ .

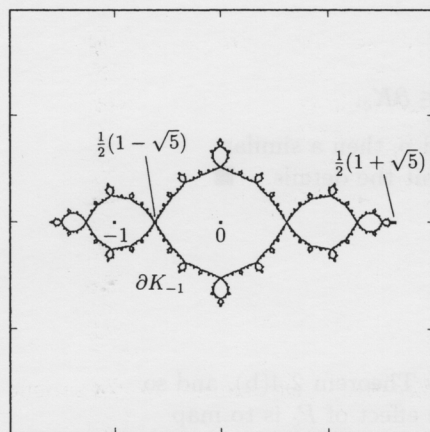


Figure 2.12  $\partial K_{-1}$

This picture suggests that the repelling fixed points  $\frac{1}{2}(1 \pm \sqrt{5})$  lie on the boundary of  $K_{-1}$ , whereas the super-attracting 2-cycle  $0, -1$  lies in the interior of  $K_{-1}$ . A similar phenomenon occurred for  $P_0$  in the solution to Problem 2.7(b). The super-attracting fixed point  $0$  of  $P_0$  lies in the interior of  $K_0$ , whereas the repelling fixed point at  $1$  and the 2-cycle and the 3-cycles, which are all repelling, lie on the boundary of  $K_0$ . In fact, the following general result holds.

See Problem 2.8(b), for example.

**Theorem 2.5** Let  $\alpha$  be a periodic point of the function  $P_c$ .

- (a) If  $\alpha$  is attracting, then  $\alpha$  is an interior point of  $K_c$ .
- (b) If  $\alpha$  is repelling, then  $\alpha$  is a boundary point of  $K_c$ .

The definitions of interior point and boundary point are given in Unit A3, Subsection 5.1.

## Proof

- (a) Suppose that  $\alpha$  is an attracting periodic point of  $P_c$ , with period  $p$ . Then  $\alpha$  is an attracting fixed point of the  $p$ th iterate  $P_c^p$ . Hence, by Theorem 1.1, there is an open disc with centre  $\alpha$  whose points are attracted to  $\alpha$  under iteration of  $P_c^p$ . These points therefore do not escape to  $\infty$  under iteration of  $P_c$ , and so they must lie in  $K_c$ . Hence  $\alpha$  is an interior point of  $K_c$ .
- (b) First suppose that  $\alpha$  is a repelling fixed point, so that

$$P_c(\alpha) = \alpha \quad \text{and} \quad |P'_c(\alpha)| > 1.$$

This proof may be omitted on a first reading.

Since  $\alpha \in K_c$ , we need to show that  $\alpha$  is *not* an interior point of  $K_c$ . If it were an interior point, then we could choose an open disc  $\{z : |z - \alpha| < r\}$  lying in  $K_c$ . In that case

$$P_c^n(z) \in K_c, \quad \text{for } |z - \alpha| = \frac{1}{2}r \text{ and } n = 1, 2, \dots,$$

by Theorem 2.3(d), and hence

$$|P_c^n(z)| \leq r_c, \quad \text{for } |z - \alpha| = \frac{1}{2}r \text{ and } n = 1, 2, \dots,$$

by Theorem 2.3(a). Now we apply Cauchy's Estimate to each of the polynomial functions  $P_c^n$  to deduce that

$$|(P_c^n)'(\alpha)| \leq \frac{r_c}{\frac{1}{2}r} = \frac{2r_c}{r}, \quad \text{for } n = 1, 2, \dots \quad (2.11)$$

By the Chain Rule, as in the proof of Theorem 2.4,

$$\begin{aligned} (P_c^n)'(\alpha) &= P_c'(P_c^{n-1}(\alpha)) \times \dots \times P_c'(P_c(\alpha)) \times P_c'(\alpha) \\ &= (P_c'(\alpha))^n, \end{aligned}$$

since, by assumption,  $\alpha$  is a fixed point of  $P_c$ . Because  $|P_c'(\alpha)| > 1$ , this implies that the sequence

$$|(P_c^n)'(\alpha)| = |P_c'(\alpha)|^n, \quad n = 1, 2, \dots,$$

tends to  $\infty$ , contrary to Estimate (2.11). Thus  $\alpha \in \partial K_c$ .

If  $\alpha$  is a repelling periodic point of  $P_c$ , with period  $p$ , then a similar argument applies (with  $P_c^p$  rather than  $P_c$ ); we omit the details. ■

## Remarks

1 Notice in part (a) that each point of the cycle

$$\alpha, P_c(\alpha), P_c^2(\alpha), \dots, P_c^{p-1}(\alpha),$$

is an attracting periodic point of  $P_c$ , with period  $p$ , by Theorem 2.4(b), and so each point of this cycle is an interior point of  $K_c$ . The effect of  $P_c$  is to map any point  $z$  near  $\alpha$  to a point  $P_c(z)$  near  $P_c(\alpha)$ , and so on, round and round the cycle (see Figure 2.13, in which, for convenience, each disc has the same radius).

If the sequence

$$z, P_c^p(z), P_c^{2p}(z), \dots$$

tends to  $\alpha$ , then the sequence

$$P_c(z), P_c^{p+1}(z), P_c^{2p+1}(z), \dots$$

tends to  $P_c(\alpha)$ , by the continuity of  $P_c$  at  $\alpha$ .

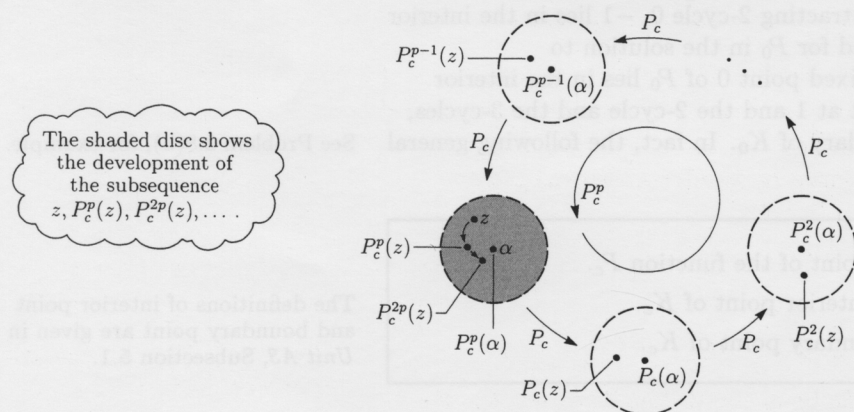


Figure 2.13

In this way the sequence  $\{P_c^n(z)\}$  forms itself into  $p$  convergent subsequences, each converging to a point of the attracting  $p$ -cycle. In fact it can be shown that *every* interior point of  $K_c$  will be attracted in this way to the attracting  $p$ -cycle. See Figure 2.14, which shows an interior point  $z_0$  of  $K_{-1}$  being attracted to the super-attracting 2-cycle  $0, -1$ .

2 In view of Theorem 2.5, it is natural to ask where in  $K_c$  do any *indifferent* periodic points of  $P_c$  lie. The answer is that it depends in a rather complicated way on the multiplier of the periodic point. This multiplier has modulus 1, and so it is of the form  $e^{2\pi ia}$ , where  $0 \leq a < 1$ . It can be shown that if  $a$  is a

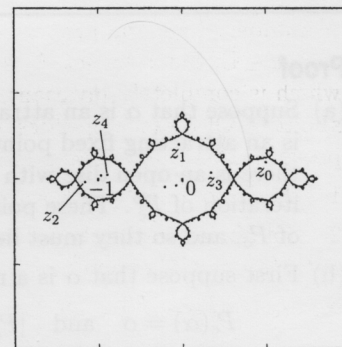


Figure 2.14  
 $\partial K_{-1}$  and  $z_n = P_{-1}^n(z_0)$



rational number, then the periodic point lies on  $\partial K_c$ , whereas if  $a$  is an irrational number, then the periodic point *usually* lies in the interior of  $K_c$ .

**3** If we know that the set  $K_c$  has no interior points, for a given value of  $c$ , then Theorem 2.5(a) tells us that  $P_c$  can have no attracting periodic points. For example,  $K_{-2} = \{x + iy : |x| \leq 2, y = 0\}$  has no interior points and so  $P_{-2}(z) = z^2 - 2$  has no attracting periodic points, a fact which is not immediately obvious!

Problem 2.5

## 2.4 The Julia set of $P_c$

We saw in the previous subsection that the boundary of the keep set  $K_c$  (which is also the boundary of the escape set  $E_c$ ) contains all the repelling periodic points of  $P_c$ . This set is of particular interest because it forms the 'watershed' between those points which escape to  $\infty$  under iteration of  $P_c$  and those which are kept (see Figure 2.15). It is called the *Julia set* of  $P_c$ , in honour of the mathematician who first studied it in detail.

This section is intended for reading only.

Gaston Julia (1893–1978), a French mathematician, developed the basic theory of the iteration of analytic functions while recovering from wounds received in World War I, and won a prize for this work from the Parisian Academy of Sciences. He was a professor at the Sorbonne, and worked on many aspects of complex analysis.

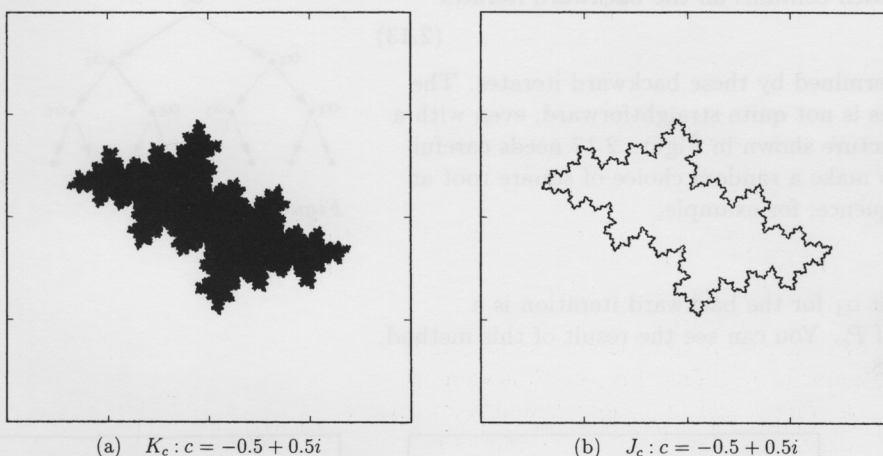


Figure 2.15

**Definition** The **Julia set**  $J_c$  of  $P_c$  is the boundary of  $K_c$ .

As noted earlier, the set  $K_c$  has no holes in it and so  $K_c$  can be thought of as  $J_c$  together with the 'inside' of  $J_c$ . Hence  $K_c$  is often called the *filled-in Julia set*. For example,

$$K_0 = \{z : |z| \leq 1\}, \quad \text{so } J_0 = \partial K_0 = \{z : |z| = 1\},$$

and

$$K_{-2} = \{x + iy : |x| \leq 2, y = 0\}, \quad \text{so } J_{-2} = \partial K_{-2} = K_{-2}.$$

A number of general properties of  $J_c$  can be deduced from Theorem 2.3. For each  $c \in \mathbb{C}$ , the Julia set  $J_c$  is a non-empty compact subset of  $\{z : |z| \leq r_c\}$ , which is completely invariant under  $P_c$  and symmetric under rotation by  $\pi$  about 0. Of these properties, only the complete invariance is a little tricky to prove, and this holds because of the following facts:

- 1  $K_c$  is completely invariant (Theorem 2.3(d));
- 2  $\text{int } K_c$  is completely invariant (by the continuity of  $P_c$  and the Open Mapping Theorem);
- 3  $J_c = \partial K_c = K_c - \text{int } K_c$ .

Notice that if  $K_c$  has no interior points then  $J_c = K_c$ .

Unit C2, Theorem 3.1

The definition of  $J_c$  could be used to plot  $J_c$ , but there are also other methods. We have already seen that  $J_c$  contains all the repelling periodic points of  $P_c$ , and so a knowledge of a number of these points gives us some information about the shape of  $J_c$ . In fact it can be shown that

$J_c$  is the smallest closed set which contains all the repelling periodic points of  $P_c$ . (2.12)

Thus the shape of  $J_c$  is entirely determined by these points. However, calculating a large number of the repelling periodic points would be a difficult task in practice.

A more satisfactory method of plotting  $J_c$  is to use the complete invariance of this set under  $P_c$ . This complete invariance tells us that if  $\alpha_1 \in J_c$ , then the solutions of the equation  $P_c(z) = \alpha_1$ , that is,  $z^2 + c = \alpha_1$ , also lie in  $J_c$ . This equation has the solutions  $\pm\sqrt{\alpha_1 - c}$ , which are two new points,  $\alpha_2, \alpha_3$  say, of  $J_c$  (see Figure 2.16). Now, however, we can repeat this process with  $\alpha_2, \alpha_3$  instead of  $\alpha_1$  to obtain four new points  $\alpha_4, \alpha_5, \alpha_6, \alpha_7$  in  $J_c$ . This process, which is illustrated schematically in Figure 2.17, is known as **backward iteration**. It can be shown that

$J_c$  is the smallest closed set which contains all the backward iterates of any given point of  $J_c$ . (2.13)

Thus the shape of  $J_c$  is entirely determined by these backward iterates. The calculation of such backward iterates is not quite straightforward, even with a computer, because the tree-like structure shown in Figure 2.17 needs careful handling. A common short cut is to make a random choice of square root at each level and plot the resulting sequence; for example,

$\alpha_1, \alpha_3, \alpha_6, \alpha_{12}, \dots$

A convenient choice of starting point  $\alpha_1$  for the backward iteration is a repelling or indifferent fixed point of  $P_c$ . You can see the result of this method, for several values of  $c$ , in Figure 2.18.

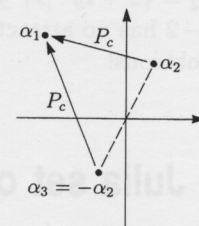


Figure 2.16

Exceptional cases occur when  $\alpha_1 = c$ , or when  $\alpha_1$  is a fixed point of  $P_c$ . In these cases, there is only one new point.

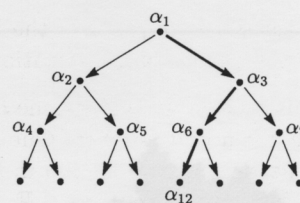


Figure 2.17

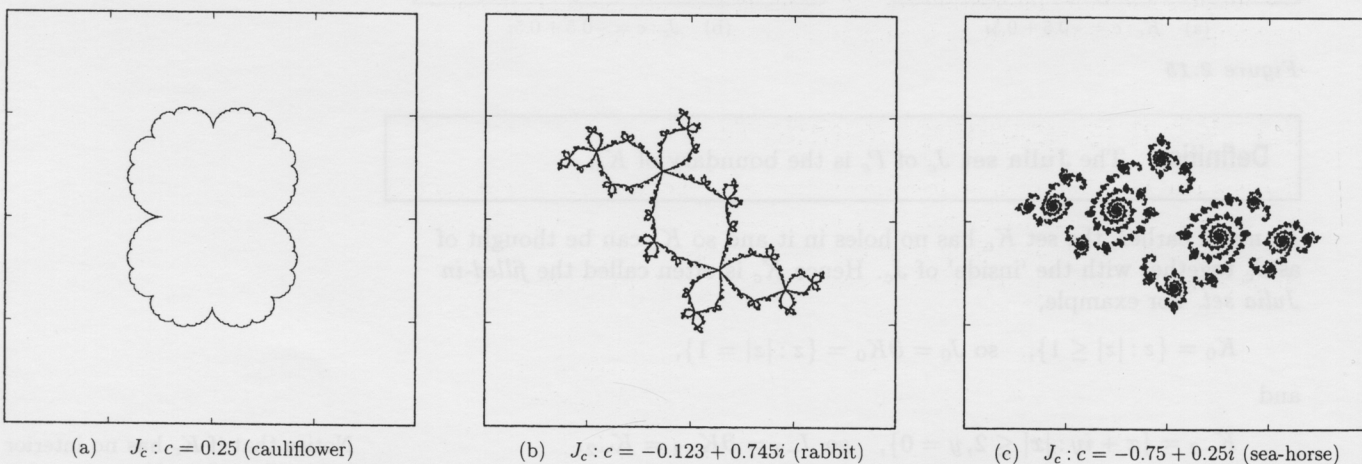


Figure 2.18

## Remarks

1 Property (2.12) implies that near every point of  $J_c$ , we can find repelling periodic points. The effect of these repelling periodic points is to make the behaviour of the function  $P_c$  on  $J_c$  extremely unstable, in the sense that points near together on  $J_c$  tend to be pushed apart under the iteration of  $P_c$ . Such behaviour is often described as *chaotic*.

By contrast the behaviour of  $P_c$  on  $\mathbb{C} - J_c$  is *stable*; that is, points close together in  $\mathbb{C} - J_c$  behave in essentially the same way under iteration of  $P_c$ . This distinction between stable and unstable behaviour can be used to define the notion of a Julia set for *any* entire function or rational function.



2 Julia sets display a remarkable ‘self-similarity’ property. For each  $c$ , the shape of any part of  $J_c$  appears to be repeated all over  $J_c$  and is seen even when we zoom in closer and closer to  $J_c$ . This is a consequence of the complete invariance of  $J_c$  together with Property (2.13) of  $J_c$ . As a result, Julia sets are often described as *fractals*. The name fractal was introduced by B. Mandelbrot (of whom, more later) in 1975 to describe a type of set which is extremely irregular, and yet has an underlying structure that can be seen under successive magnifications of the set. The exact definition of a ‘fractal’ is still under discussion, but it has to do with certain methods of measuring the *dimension* of a set which may give non-integer answers!

### 3 GRAPHICAL ITERATION

After working through this section, you should be able to:

- use *graphical iteration* to determine the behaviour of real iteration sequences;
- describe properties of the keep sets  $K_c$ , for  $c \in \mathbb{R}$ , which can be obtained by using graphical iteration.

In this section we make some observations about the nature of the keep sets  $K_c$  when  $c$  is a *real* number. One simple observation is that if  $c$  is real, then  $K_c$  is symmetric under reflection in the real axis (see, for example,  $K_1$  in Figure 2.4). This holds because, for  $c \in \mathbb{R}$ ,

$$P_c(\bar{z}) = \bar{z}^2 + c = \overline{z^2 + c} = \overline{P_c(z)}, \quad \text{for } z \in \mathbb{C},$$

so that, for  $c \in \mathbb{R}$ ,

$$P_c^n(\bar{z}) \rightarrow \infty \text{ as } n \rightarrow \infty \iff P_c^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty;$$

thus,  $\bar{z} \in K_c$  if and only if  $z \in K_c$ .

We obtain some more interesting results by using a technique called graphical iteration, which applies only to the iteration of *real* functions.

#### 3.1 What is graphical iteration?

If  $f$  is a real function, then an iteration sequence of the form  $x_{n+1} = f(x_n)$  can be represented graphically by using the two graphs  $y = f(x)$  and  $y = x$  plotted together, as in Figure 3.1. Note that any point where  $y = f(x)$  meets  $y = x$  corresponds to a fixed point of  $f$  (for example, the point  $a$  in Figure 3.1).

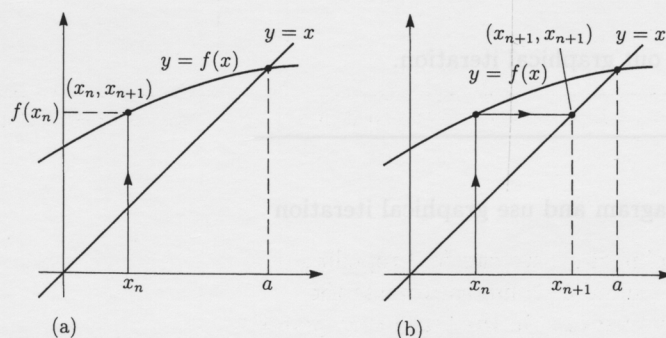


Figure 3.1

These two diagrams illustrate a two-stage process for finding geometrically the position on the  $x$ -axis of the term  $x_{n+1}$ , given the position of the term  $x_n$ :

- draw a vertical line to meet  $y = f(x)$  at  $(x_n, f(x_n)) = (x_n, x_{n+1})$ ;
- draw a horizontal line to meet  $y = x$  at  $(x_{n+1}, x_{n+1})$ .

Given any initial term  $x_0$ , we can apply the above process repeatedly to construct the sequence  $\{x_n\}$  geometrically, and thus obtain information about the behaviour of  $\{x_n\}$ . For example, with the function  $f$  in Figure 3.1 and with  $x_0 = 0$ , we obtain the behaviour illustrated in Figure 3.2, which strongly suggests that  $\{x_n\}$  tends to the fixed point  $a$ .

In the following example we carry out graphical iteration with a particular function  $f$ .

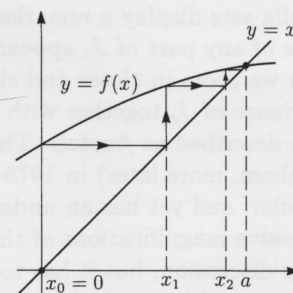


Figure 3.2

### Example 3.1

Let  $f(x) = \frac{1}{2}x + 1$ .

- (a) Plot  $y = f(x)$  and  $y = x$  on the same diagram and use graphical iteration to plot the iteration sequences

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \dots,$$

with  $x_0 = 0$  and  $x_0 = 3$ .

- (b) Describe the behaviour of each of these sequences  $\{x_n\}$  and check that your answer agrees with the solution to Problem 1.9(b)(i).

### Solution

- (a) The graphs  $y = f(x)$  and  $y = x$  are plotted in Figure 3.3. They meet at the point  $(2, 2)$ , which corresponds to 2, the only fixed point of  $f$ . The sequences  $x_{n+1} = f(x_n)$ ,  $n = 0, 1, 2, \dots$ , with  $x_0 = 0$  and  $x_0 = 3$ , are also plotted.

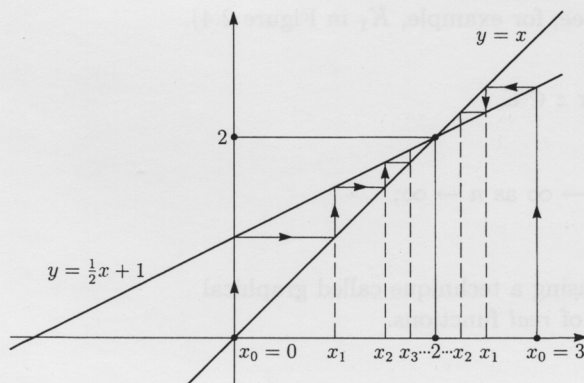


Figure 3.3

- (b) For both values of  $x_0$ , the figure strongly suggests that  $x_n \rightarrow 2$  as  $n \rightarrow \infty$ . In Problem 1.9(b)(i) we found that if  $|a| < 1$ , then the iteration sequence  $z_{n+1} = az_n + b$ ,  $n = 0, 1, 2, \dots$ , converges to (the fixed point)  $\alpha = b/(1 - a)$  for all  $z_0$ . Here we have  $a = \frac{1}{2}$ ,  $b = 1$  and  $\alpha = 2$ , so our answer agrees with this result. ■

The next problem gives you a chance to try out graphical iteration.

### Problem 3.1

Let  $f(x) = -2x + 1$ .

- (a) Plot  $y = f(x)$  and  $y = x$  on the same diagram and use graphical iteration to plot the iteration sequences

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \dots,$$

with  $x_0 = 0$  and  $x_0 = \frac{1}{3}$ .

- (b) Describe the behaviour of each of these sequences  $\{x_n\}$  and check that your answer agrees with the solution to Problem 1.9(b)(i).



## 3.2 Real quadratic iteration sequences

We now apply graphical iteration to real quadratic iteration sequences of the form

$$x_{n+1} = P_c(x_n) = x_n^2 + c, \quad n = 0, 1, 2, \dots,$$

where  $c$  is real. A great deal can be said about sequences of this special form, but we confine attention to a few basic results which throw some light on the corresponding keep sets  $K_c$ .

To illustrate the method, we consider the case  $c = 0$ . The graphs  $y = P_0(x) = x^2$  and  $y = x$  meet at  $(0, 0)$  and  $(1, 1)$ , corresponding to the fixed points 0, 1 of  $P_0$ , and the iterations plotted in Figure 3.4 indicate that if  $|x| \leq 1$  then

$$0 \leq P_0^n(x) \leq 1, \quad \text{for } n = 1, 2, \dots$$

On the other hand, if  $|x| > 1$ , then

$$P_0^n(x) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

These results show that the part of the keep set  $K_0 = \{z : |z| \leq 1\}$  on the real axis is the closed interval  $[-1, 1]$ , as expected.

Now the only values of  $c$  for which we know  $K_c$  explicitly are  $c = 0$  and  $c = -2$ . For other *real* values of  $c$ , we may expect that graphical iteration will give new information about  $K_c$ . To see what kind of information can be obtained in this way, try the following problem.

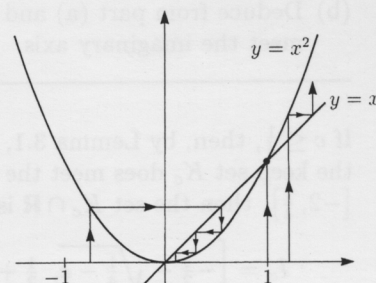


Figure 3.4

### Problem 3.2

- Plot  $y = x^2 + 1$  and  $y = x$  on the same diagram and apply graphical iteration to the sequence  $x_{n+1} = x_n^2 + 1$ ,  $n = 0, 1, 2, \dots$ , with your own choice of initial term  $x_0$ .
- Explain why the sequence  $\{x_n\}$  tends to infinity.
- What do you deduce about the set  $K_1$ ?

The solution to Problem 3.2 suggests that the presence or absence of real fixed points of  $P_c(x) = x^2 + c$  is of fundamental importance to the behaviour of iteration sequences of the form  $x_{n+1} = x_n^2 + c$ . Since the fixed points of  $P_c$  are  $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$ , the following lemma is evident.

**Lemma 3.1** If  $c \in \mathbb{R}$ , then the function  $P_c$  has

- no real fixed points if  $c > \frac{1}{4}$ ;
- the single fixed point  $\frac{1}{2}$ , if  $c = \frac{1}{4}$ ;
- the two real fixed points  $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$ , if  $c < \frac{1}{4}$ .

Part (a) of Lemma 3.1 shows that if  $c > \frac{1}{4}$ , then the graph  $y = x^2 + c$  lies entirely above  $y = x$  (see Figure 3.5). It follows by graphical iteration that if  $c > \frac{1}{4}$ , then

$$P_c^n(x) \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \text{for all } x \in \mathbb{R}. \quad (3.1)$$

Thus, if  $c > \frac{1}{4}$ , then all real values of  $x$  escape to  $\infty$  under iteration of  $P_c$ , and so no real values of  $x$  belong to the keep set  $K_c$ . This gives the following result.

**Theorem 3.1** If  $c > \frac{1}{4}$ , then  $K_c \cap \mathbb{R} = \emptyset$ .

Problem 2.3

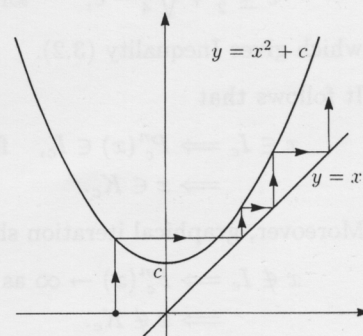


Figure 3.5  $c > \frac{1}{4}$

An algebraic proof of (3.1) is as follows.

If  $x_{n+1} = x_n^2 + \frac{1}{4} + \varepsilon$ , where  $\varepsilon > 0$ , then

$$x_{n+1} - x_n = \left(x_n - \frac{1}{2}\right)^2 + \varepsilon \geq \varepsilon;$$

hence

$$x_n \geq x_0 + n\varepsilon$$

and so  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

It follows, by Theorem 2.3, parts (c) and (e), that if  $c > \frac{1}{4}$ , then  $K_c$  must have points in both the upper and lower open half-planes. Thus, for these values of  $c$ , the set  $K_c$  is in at least two pieces, and so is not connected. Moreover, the plot of  $K_1$  in Figure 2.4 suggests that the set  $K_1$  does not meet the imaginary axis. We ask you to prove this for  $K_c, c > \frac{1}{4}$ , in the next problem.

### Problem 3.3

- (a) Show that if  $c$  is real and  $y$  is real, then  $P_c(iy)$  is real.  
 (b) Deduce from part (a) and Theorem 3.1 that if  $c > \frac{1}{4}$ , then  $K_c$  does not meet the imaginary axis.

If  $c \leq \frac{1}{4}$ , then, by Lemma 3.1,  $P_c$  has either one or two real fixed points and so the keep set  $K_c$  does meet the real axis. It turns out that if  $c$  lies in the interval  $[-2, \frac{1}{4}]$ , then the set  $K_c \cap \mathbb{R}$  is precisely equal to the symmetric closed interval

$$I_c = \left[ -\frac{1}{2} - \sqrt{\frac{1}{4} - c}, \frac{1}{2} + \sqrt{\frac{1}{4} - c} \right].$$

Note that  $I_0 = [-1, 1]$  and  $I_{-2} = [-2, 2]$ , as expected.

**Theorem 3.2** If  $-2 \leq c \leq \frac{1}{4}$ , then  $K_c \cap \mathbb{R} = I_c$ .

**Proof** The graphs  $y = P_c(x)$  and  $y = x$  are shown in Figure 3.6, together with a square  $S$  with sides parallel to the axes, which meets both axes in the interval  $I_c$ . The key to the proof is the observation that if  $-2 \leq c \leq \frac{1}{4}$ , then the points of the graph  $y = P_c(x)$ , for  $x \in I_c$ , lie in  $S$ . If  $0 \leq c \leq \frac{1}{4}$ , then this is evident because  $y = P_c(x)$  does not extend below the  $x$ -axis. If  $-2 \leq c \leq 0$ , then we need to show that the lowest point of  $y = P_c(x)$  does not lie below the bottom edge of the square  $S$ , that is,

$$c \geq -\frac{1}{2} - \sqrt{\frac{1}{4} - c}, \quad \text{for } -2 \leq c \leq 0. \quad (3.2)$$

This inequality can be verified directly or we can use Figure 2.2. This shows that

$$|c| \leq r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}, \quad \text{for } 0 \leq |c| \leq 2,$$

and hence that

$$-c \leq \frac{1}{2} + \sqrt{\frac{1}{4} - c}, \quad \text{for } -2 \leq c \leq 0,$$

which gives Inequality (3.2).

It follows that

$$\begin{aligned} x \in I_c &\implies P_c^n(x) \in I_c, \quad \text{for } n = 1, 2, \dots, \\ &\implies x \in K_c. \end{aligned}$$

Moreover, graphical iteration shows that

$$\begin{aligned} x \notin I_c &\implies P_c^n(x) \rightarrow \infty \text{ as } n \rightarrow \infty \\ &\implies x \notin K_c. \end{aligned}$$

Thus  $K_c \cap \mathbb{R} = I_c$ . ■

If  $c = -2$ , then we know that

$$K_{-2} = \{x + iy : |x| \leq 2, y = 0\} = I_{-2}.$$

However, it can be shown that for other values of  $c$  in the interval  $[-2, \frac{1}{4}]$ , the keep set  $K_c$  does not lie entirely on the real axis.

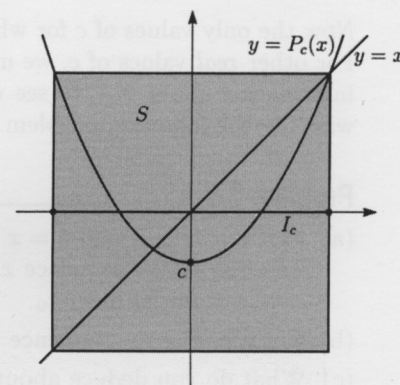


Figure 3.6



For  $c < -2$ , the keep set  $K_c$  has a more interesting intersection with the real axis, which is described in the next result.

**Theorem 3.3** If  $c < -2$ , then the set  $K_c \cap \mathbb{R}$  consists of the closed interval  $I_c$  from which a sequence of disjoint, non-empty, open subintervals of  $I_c$  has been removed. In particular,  $0 \notin K_c$ .

**Proof** Let  $S$  be the square used in the proof of Theorem 3.2, which meets the axes in the interval  $I_c$ . First note that, by graphical iteration,

$$P_c^n(x) \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \text{for } x \in \mathbb{R} - I_c, \quad (3.3)$$

so that  $K_c \cap \mathbb{R} \subseteq I_c$ .

Now, since  $c < -2$ , the lowest point on  $y = P_c(x)$  lies below  $S$ , as in Figure 3.7. Therefore, the set  $A_0$  of points in  $I_c$  which escape from  $I_c$  after exactly one iteration of  $P_c$ ,

$$A_0 = \{x \in I_c : P_c(x) \notin I_c\},$$

is an open subinterval of  $I_c$  with centre 0 (see Figure 3.7). In view of (3.3), it follows that

$$P_c^n(x) \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \text{for } x \in A_0,$$

and so the points of  $A_0$  do not lie in  $K_c$ . In particular,  $0 \notin K_c$ .

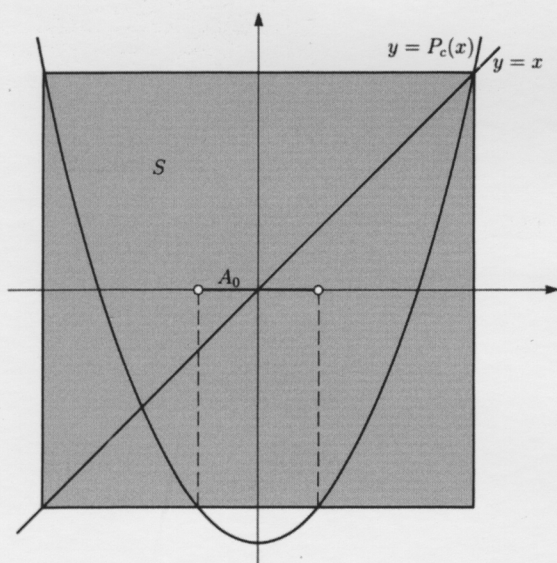


Figure 3.7

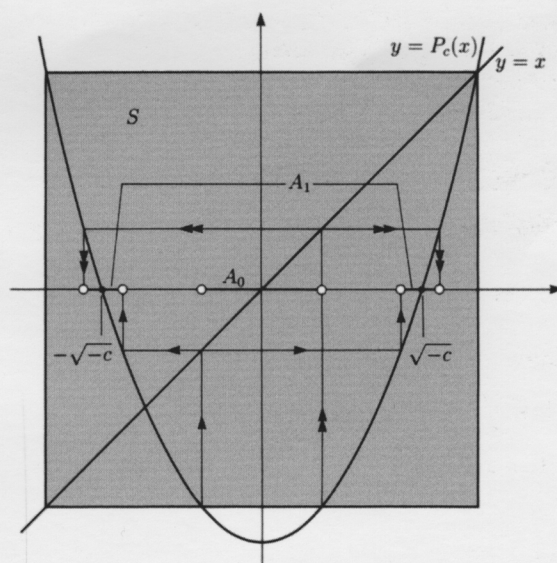


Figure 3.8

We now consider the set  $A_1$  of points in  $I_c$  which remain in  $I_c$  for exactly one iteration of  $P_c$ :

$$\begin{aligned} A_1 &= \{x \in I_c : P_c(x) \in I_c, \text{ but } P_c^2(x) \notin I_c\} \\ &= \{x \in I_c : P_c(x) \in A_0\}. \end{aligned}$$

The set  $A_1$  (see Figure 3.8) consists of two open subintervals of  $I_c$  which are positioned symmetrically on either side of 0 and contain the points  $\pm\sqrt{-c}$ , which are the zeros of  $P_c$ .

More generally, we consider the set  $A_n$  of points in  $I_c$  which remain in  $I_c$  for exactly  $n$  iterations of  $P_c$ , defined inductively as follows:

$$A_n = \{x \in I_c : P_c(x) \in A_{n-1}\}, \quad \text{for } n = 1, 2, \dots$$

The construction lines with single and double arrowheads in Figure 3.8 indicate how the endpoints of the two open subintervals, which comprise  $A_1$ , are found.

Since each open interval in  $A_{n-1}$  gives rise to two open intervals in  $A_n$ , it follows that the set  $A_n$  consists of  $2^n$  disjoint open subintervals of  $I_c$ . Now, any point of  $I_c$  which escapes to  $\infty$  under iteration of  $P_c$  must lie in exactly one of the sets  $A_n$ . Thus the sets  $A_n$  are disjoint and

$$K_c \cap \mathbb{R} = I_c - (A_0 \cup A_1 \cup \dots),$$

which gives the required structure. ■

### Remarks

**1** The set  $K_c \cap \mathbb{R} = I_c - (A_0 \cup A_1 \cup \dots)$  is infinite because it contains all the endpoints of all the intervals which comprise  $A_n$ , for  $n = 0, 1, 2, \dots$ . Actually, there are infinitely many points in  $K_c \cap \mathbb{R}$  which are *not* endpoints of this type, but these are harder to identify.

**2** It can be shown that if  $c < -2$ , then  $K_c \subseteq \mathbb{R}$  (as is the case for  $c = -2$ ).

### Problem 3.4

Show that if  $c < -2$ , then  $K_c$  does not meet the imaginary axis.

(Hint: Problem 3.3(a) and Figure 3.7 should help.)

Problem 3.4 shows that if  $c < -2$ , then the set  $K_c$  is in at least two pieces, and so (as when  $c > \frac{1}{4}$ ) it is not connected. We pursue the question of the connectedness of  $K_c$  in the next section.

This can be *proved* by using Mathematical Induction.



## 4 THE MANDELBROT SET



After working through this section, you should be able to:

- understand the definition of the *Mandelbrot set*  $M$ ;
- use the Fatou–Julia Theorem and its corollaries to determine whether certain points lie in  $M$ ;
- appreciate how a computer may be used to plot  $M$ ;
- show that certain points  $c$  lie in  $M$  because the corresponding function  $P_c$  has an *attracting cycle*;
- appreciate where certain periodic regions of the Mandelbrot set are located by making use of *saddle-node bifurcations* and *period-multiplying bifurcations*.

### 4.1 What is the Mandelbrot set?

In Section 2 we gave computer-generated plots of a number of keep sets  $K_c$  for quadratic functions of the form  $P_c(z) = z^2 + c$ . The shapes of these sets are remarkably varied, but they can be classified into two distinct types: those which are ‘all in one piece’ (for example,  $K_0$  and  $K_{-2}$ ) and those which are ‘in more than one piece’ (for example,  $K_1$ ). The mathematical name for a set which is ‘all in one piece’ is *connected*.

Earlier in the course we introduced *pathwise connectedness*, in order to define the key concept of a *region*. It is clear that the sets  $K_0$  and  $K_{-2}$  are both pathwise connected, but it is not so evident that the complicated set  $K_{-1}$  is pathwise connected, even though it does appear to be in one piece. We find it convenient to introduce the following more general notion of connectedness.

Unit A3, Section 4

**Definitions** A set  $A$  is **disconnected** if there are disjoint open sets  $G_1$  and  $G_2$  such that

$$A \cap G_1 \neq \emptyset, \quad A \cap G_2 \neq \emptyset \quad \text{and} \quad A \subseteq G_1 \cup G_2.$$

A set  $A$  is **connected** if it is not disconnected.

Throughout the rest of this unit, this is the meaning of the word *connected*.

For example, for each real  $c > \frac{1}{4}$ , the set  $K_c$  is disconnected because, by Theorem 3.1 and Problem 2.3(a), it does not meet the real axis, but it does have points in both the upper and lower open half-planes. Thus, for  $c > \frac{1}{4}$ , the definition of disconnected is satisfied with  $A = K_c$ ,  $G_1 = \{z : \text{Im } z > 0\}$ , and  $G_2 = \{z : \text{Im } z < 0\}$ ; see Figure 4.1, where  $c = \frac{1}{2}$ .

#### Problem 4.1

Use the result of Problem 3.4 to show that  $K_c$  is disconnected for real  $c < -2$ .

It is more difficult to prove that a connected set *is* connected. This is because we need to show that it is not disconnected, that is, the open sets  $G_1, G_2$  in the above definition *do not exist*. However, it is not too difficult to show that any pathwise connected set is connected

(although we omit the details), and so the sets  $K_0$  and  $K_{-2}$  are connected. At present, we set aside the difficulty of proving that a set is connected and

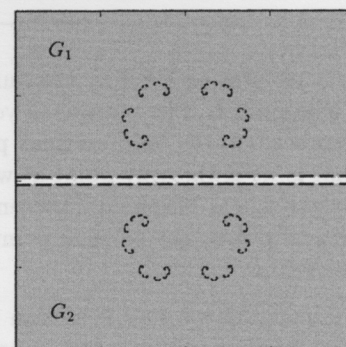


Figure 4.1  $K_{1/2}$

However a connected set need not be pathwise connected.

concentrate on the set of points  $c$  such that  $K_c$  is connected; this set is the celebrated *Mandelbrot set*.

**Definition** The **Mandelbrot set** is the set  $M$  of complex numbers  $c$  such that  $K_c$  is connected.

**Remark** This definition is often phrased in terms of the connectedness of the Julia set  $J_c$ . By using the fact that  $K_c$  has no holes in it, it can be shown that  $K_c$  is connected if and only if  $J_c$  is connected.

For example, since  $K_0$  and  $K_{-2}$  are connected, we have  $0 \in M$  and  $-2 \in M$ . On the other hand, by Problem 4.1 and the discussion preceding it, we have  $c \notin M$ , for all real  $c > \frac{1}{4}$  and  $c < -2$ .

The definition of  $M$  is typical of many in mathematics. It defines the object that we are interested in precisely, but as it stands it is difficult to work with. For most values of  $c$  we have very little idea what  $K_c$  looks like, let alone whether or not it is connected! Fortunately, however, there is a numerical method of deciding whether  $K_c$  is connected, based on the following fundamental result, the proof of which is outlined in Subsection 4.3.

### Theorem 4.1 Fatou–Julia Theorem

For any  $c \in \mathbb{C}$ ,

$$K_c \text{ is connected} \iff 0 \in K_c.$$

This result states that the keep set  $K_c$  is connected if and only if the point 0 does not escape to  $\infty$  under iteration of  $P_c$ . For example, 0 lies in both  $K_0$  and  $K_{-2}$ , which are connected, but 0 does not lie in  $K_1$ , which is disconnected.

To see the power of Theorem 4.1, try the following problem, which characterizes that part of the Mandelbrot set which lies on the real axis.

### Problem 4.2

- Use Theorem 4.1 and Theorem 3.2 to show that if  $c \in [-2, \frac{1}{4}]$ , then  $c \in M$ .
- Deduce that  $M \cap \mathbb{R} = [-2, \frac{1}{4}]$ .

It is natural to ask why the number 0 appears in this special way in Theorem 4.1. The reason, as you will see in Subsection 4.3, is that the number 0 is the only **critical point** of each of the functions  $P_c(z) = z^2 + c$ ; that is, it is the only point at which each of the derivatives  $P'_c(z) = 2z$  vanishes. By the Local Mapping Theorem, an analytic function fails to be one-one near a critical point, and so such points play a significant role in the function's behaviour.

By using Theorem 2.3, we can turn the condition  $0 \in K_c$  in Theorem 4.1 into a numerical condition which is easier to check. By Theorem 2.3(d),

$$0 \in K_c \implies P_c^n(0) \in K_c, \text{ for } n = 0, 1, 2, \dots,$$

and so, by Theorem 2.3(a),

$$0 \in K_c \implies |P_c^n(0)| \leq r_c, \text{ for } n = 0, 1, 2, \dots, \quad (4.1)$$

where  $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$ .

Benoit Mandelbrot was born in Warsaw in 1924. His initial training as a mathematician took place at the Ecole Polytechnique, Paris. He started working for IBM in 1958 and became an IBM Fellow in 1974, at the Watson Research Institute in New York State.

Pierre Fatou (1878–1929) was a French astronomer, although he trained as a mathematician. In addition to his work on iteration theory, he proved an important result about the boundary behaviour of complex functions. Fatou and Julia (see page 27) proved (independently) a more general version of this result in 1918–19. After their pioneering work in complex iteration, there were only a few other developments in this field until the explosion of interest in the 1980s, sparked off by the use of computers.

Unit C2, Theorem 3.2



Since  $P_c(0) = c$ , the right-hand side of Implication (4.1) shows that  $|c| \leq r_c$  and hence, from Figure 2.2, that  $r_c \leq 2$ . Thus

$$0 \in K_c \implies |P_c^n(0)| \leq 2, \text{ for } n = 1, 2, \dots \quad (4.2)$$

Now, if  $c$  satisfies the inequalities on the right-hand side of Implication (4.2), then the sequence  $\{P_c^n(0)\}$  is bounded and so  $0 \in K_c$ , by the definition of  $K_c$ . Hence, by the definition of  $M$  and Theorem 4.1,

$$\begin{aligned} c \in M &\iff K_c \text{ is connected} \\ &\iff 0 \in K_c \\ &\iff |P_c^n(0)| \leq 2, \text{ for } n = 1, 2, \dots, \end{aligned}$$

and so we obtain the following corollary to Theorem 4.1.

**Corollary 1** The Mandelbrot set  $M$  can be specified as follows:

$$M = \{c : |P_c^n(0)| \leq 2, \text{ for } n = 1, 2, \dots\}.$$

**Remark** Note that if  $c \notin M$ , then  $0 \notin K_c$ , by Theorem 4.1, and so  $P_c^n(0) \rightarrow \infty$  as  $n \rightarrow \infty$ .

Corollary 1 provides a numerical criterion for determining whether or not a point  $c$  belongs to  $M$ . For any given  $c$ , we simply compute the terms of the sequence  $\{P_c^n(0)\}$ , which are

$$c, c^2 + c, (c^2 + c)^2 + c, \dots,$$

and try to decide whether *all* these terms lie in  $\{z : |z| \leq 2\}$ .

For example, if  $c = 0$ , then  $\{P_c^n(0)\}$  is the constant sequence

$$0, 0, 0, \dots;$$

since all these terms lie in  $\{z : |z| \leq 2\}$ , we deduce by Corollary 1 that  $0 \in M$ . On the other hand, if  $c = 1$ , then the terms of  $\{P_c^n(0)\}$  are

$$1, 2, 5, 26, \dots;$$

since these terms do not all lie in  $\{z : |z| \leq 2\}$ , we deduce by Corollary 1 that  $1 \notin M$ .

### Problem 4.3

Use Corollary 1 to determine which of the following points lie in  $M$ .

- (a)  $c = -2$     (b)  $c = 1 + i$     (c)  $c = i$     (d)  $c = \sqrt{2}i$

Corollary 1 makes it possible to use a computer to plot an approximation to  $M$ . A naive algorithm involves checking the inequality

$$|P_c^n(0)| \leq 2 \quad (4.3)$$

for a large number of points  $c$ , and for  $n = 1, 2, \dots, N$ , where  $N$  is a suitably large positive integer. If Inequality (4.3) is false for some  $n$ , then the corresponding  $c$  lies outside  $M$ , but if it is true for  $n = 1, 2, \dots, N$ , then  $c$  must be in  $M$  or 'close to  $M$ '. The set  $M$  was first plotted in 1979 using an algorithm of this kind by Mandelbrot (who had previously been plotting Julia sets).

$$\text{For } n = 1, 2, \dots,$$

$$P_c^{n+1}(0) = (P_c^n(0))^2 + c.$$

A plot of the 'periodic regions' of  $M$  (see Subsection 4.2) was made in 1978 by R. Brooks and J. P. Matelski. They had encountered the iteration of quadratic functions while studying various groups of Möbius transformations.

A rendering of the set  $M$  using this approach is shown in Figure 4.2. As you can see, the set appears to be very complicated, consisting of many 'blobs' (the main one of which is bounded by a cardioid), which Mandelbrot called *atoms*, arranged in a highly organized manner. Some of these atoms are stuck together to form a complex *molecule*, whereas others appear to float free.

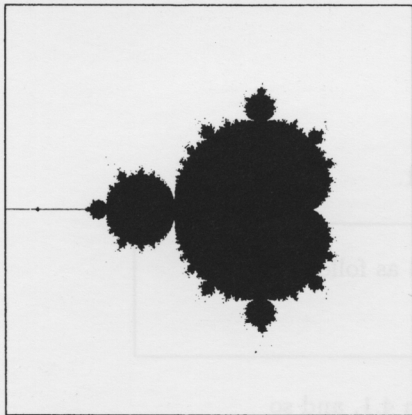


Figure 4.2  $M$  — naive algorithm

The box represented in Figure 4.2 is

$$\{c : -2 \leq \operatorname{Re} c < 1, \\ -1.5 \leq \operatorname{Im} c \leq 1.5\}.$$

Note that  $M$  is in the  $c$ -plane, often called the *parameter plane*, whereas each keep set  $K_c$  lies in the  $z$ -plane, often called the *dynamical plane*.

We can use Corollary 1 obtain a number of basic properties of  $M$ .

**Corollary 2** The Mandelbrot set  $M$

- (a) is a compact subset of  $\{c : |c| \leq 2\}$ ;
- (b) is symmetric under reflection in the real axis;
- (c) meets the real axis in the interval  $[-2, \frac{1}{4}]$ ;
- (d) has no holes in it.

Statement (d) means that  $\mathbb{C} - M$  is connected.

**Proof** First note that each term of the sequence  $\{P_c^n(0)\}$  defines a polynomial function of  $c$ . Indeed

$$\begin{aligned} P_c(0) &= c, \\ P_c^2(0) &= c^2 + c, \\ P_c^3(0) &= (c^2 + c)^2 + c = c^4 + 2c^3 + c^2 + c, \end{aligned}$$

and, in general,  $P_c^n(0)$  takes the form

$$P_c^n(0) = c^{2^{n-1}} + 2^{n-2}c^{2^{n-1}-1} + \cdots + c^2 + c, \quad \text{for } n \geq 1. \quad (4.4)$$

(a) To prove part (a), we define the sets

$$M_n = \{c : |P_c^n(0)| \leq 2\}, \quad \text{for } n = 1, 2, \dots,$$

so that  $M_1 = \{c : |c| \leq 2\}$ ,  $M_2 = \{c : |c^2 + c| \leq 2\}$ , and so on. Then, by Corollary 1,

$$M = M_1 \cap M_2 \cap \cdots,$$

and, in particular,  $M \subseteq M_1 = \{c : |c| \leq 2\}$ . Thus  $M$  is bounded.

Each of the sets  $M_n$  is closed, because its complement

$$\mathbb{C} - M_n = \{c : |P_c^n(0)| > 2\}$$

is open. Indeed, if  $|P_{c_0}^n(0)| > 2$  for some  $c_0$ , then this inequality must hold for all  $c$  in some open disc with centre  $c_0$ , by the continuity of the function  $c \mapsto |P_c^n(0)|$ . It follows that  $M$  itself must be closed, because if  $c \notin M$ , then  $c \notin M_n$  for some  $n$ , and so some open disc with centre  $c$  must lie outside  $M_n$  and hence outside  $M$ . Hence  $M$  is closed and bounded; that is,  $M$  is compact.

This proof may be omitted on a first reading.

This form of  $P_c^n(0)$  can be justified by Mathematical Induction.

See the proof of Theorem 2.3(b) for similar reasoning.



- (b) Because  $P_c^n(0)$  is a polynomial in  $c$  with real coefficients,

$$P_{\bar{c}}^n(0) = \overline{P_c^n(0)} \implies |P_{\bar{c}}^n(0)| = |P_c^n(0)|, \text{ for } n = 1, 2, \dots$$

Hence, by Corollary 1,  $\bar{c} \in M$  if and only if  $c \in M$ , and so  $M$  is symmetric under reflection in the real axis.

- (c) You proved this in Problem 4.2(b).  
 (d) The proof that  $\mathbb{C} - M$  is connected is very similar to the proof of part (f) of Theorem 2.3, using  $\{c : |c| > 2\}$  instead of  $\{z : |z| > r_c\}$  and applying the Maximum Principle to the analytic function  $c \mapsto P_c^n(0)$  instead of to  $z \mapsto P_c^n(z)$ . We omit the details. ■

It turns out that the picture of the Mandelbrot set in Figure 4.2 is misleading. Some parts of  $M$  are so thin that the naive algorithm fails to detect them. This became clear when A. Douady and J.H. Hubbard proved the following remarkable result in 1982.

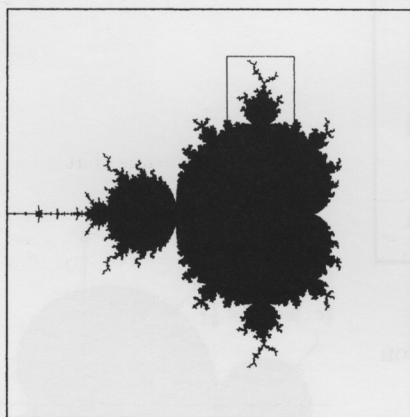
**Theorem 4.2** The Mandelbrot set is connected.

Thus it follows that all parts of  $M$  in Figure 4.2 are actually linked together. Knowing this fact, we can devise more effective algorithms for plotting better renderings of  $M$ , such as the one shown (with various enlargements corresponding to the small square boxes) in Figure 4.3.

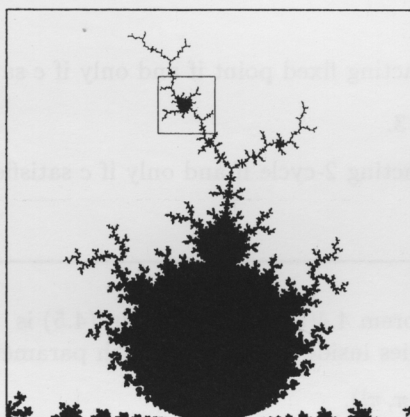
It was Douady and Hubbard who named the set  $M$  after Mandelbrot.

We make no attempt to prove Theorem 4.2.

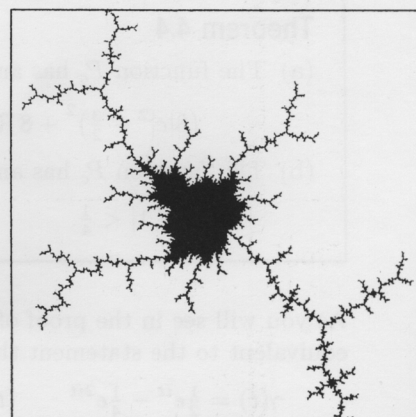
We discuss the structure of  $M$  in Subsection 4.2.



(a)



(b)



(c)

**Figure 4.3**  $M$  — showing its connectedness

To gain some insight into why the connectedness of  $M$  is so remarkable, we note that the sets  $M_n = \{c : |P_c^n(0)| \leq 2\}$ ,  $n = 1, 2, \dots$ , defined in the proof of Corollary 2, are in fact nested, that is,

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$$

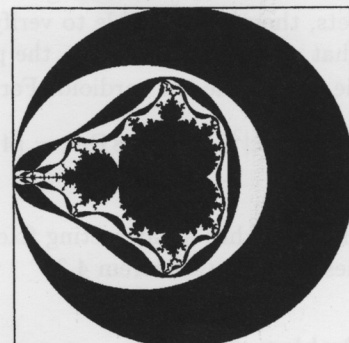
We can represent these nested sets  $M_n$  readily using a computer. See Figure 4.4, where, for the first few values of  $n$ , points of  $M_n - M_{n+1}$  are plotted black if  $n$  is odd and white if  $n$  is even. Since  $c \in M_n - M_{n+1}$  if and only if

$$|P_c^k(0)| \leq 2, \quad \text{for } k = 1, 2, \dots, n, \text{ but } |P_c^{n+1}(0)| > 2,$$

these bands measure how long the corresponding sequences  $\{P_c^n(0)\}$  remain in  $\{z : |z| \leq 2\}$ . It appears that each of the boundaries

$$\partial M_n = \{c : |P_c^n(0)| = 2\}, \quad n = 1, 2, \dots,$$

forms a simple-closed smooth path (that is, it does not break up into several pieces). This surprising fact can be shown to be equivalent to the connectedness of  $M$ .



**Figure 4.4**

By Equation (4.4),

$$\partial M_1 = \{c : |c| = 2\},$$

$$\partial M_2 = \{c : |c^2 + c| = 2\},$$

and so on.

## 4.2 Inside the Mandelbrot set

Corollary 1 of Theorem 4.1 provides a good means of showing that a given point  $c$  is *not* in the set  $M$ . This corollary is not so helpful, however, as a means of checking that  $c$  is in  $M$ . Of course, for some values of  $c$ , such as  $c = 0$  or  $c = -2$ , it is possible to check directly that  $|P_c^n(0)| \leq 2$ , for  $n = 1, 2, \dots$ , but this is usually not the case. Instead, the following general result can often be used.

**Theorem 4.3** If the function  $P_c$  has an attracting cycle, then  $c \in M$ .

**Remark** One way to prove Theorem 4.3 is to show that if  $P_c$  has an attracting cycle, then the sequence  $\{P_c^n(0)\}$  is attracted to this cycle in the way described in Remark 1 following Theorem 2.5. Since  $\{P_c^n(0)\}$  can be attracted in this way to at most one cycle, it follows that  $P_c$  has at most one attracting cycle for each value of  $c$ .

We outline another proof of Theorem 4.3 in Subsection 4.3.

Theorem 4.3 allows us to identify various key parts of  $M$ , by determining for which values of  $c$  the function  $P_c$  has an attracting  $p$ -cycle for various values of  $p$ . The solution for attracting fixed points and attracting 2-cycles is particularly elegant, as we now show.

### Theorem 4.4

(a) The function  $P_c$  has an attracting fixed point if and only if  $c$  satisfies

$$(8|c|^2 - \frac{3}{2})^2 + 8\operatorname{Re} c < 3. \quad (4.5)$$

(b) The function  $P_c$  has an attracting 2-cycle if and only if  $c$  satisfies

$$|c + 1| < \frac{1}{4}. \quad (4.6)$$

As you will see in the proof of Theorem 4.4(a), the Condition (4.5) is equivalent to the statement that  $c$  lies inside the cardioid with parametrization

$$\gamma(t) = \frac{1}{2}e^{it} - \frac{1}{4}e^{2it} \quad (t \in [-\pi, \pi]).$$

This cardioid is the boundary of the main ‘atom’ of  $M$ . Condition (4.6) means that  $c$  lies inside the open disc  $\{c : |c + 1| < \frac{1}{4}\}$  which lies immediately to the left of the cardioid; see Figure 4.5. Thus if a point  $c$  lies in one of these two sets, then it is possible to verify this using Theorem 4.4, and hence to show that  $c \in M$ . For example, the point  $c = -\frac{1}{2} + \frac{1}{2}i$ , shown in Figure 4.5, seems to lie just inside the cardioid. For this value of  $c$ ,  $|c|^2 = \frac{1}{2}$  and  $\operatorname{Re} c = -\frac{1}{2}$ , and so

$$\begin{aligned} (8|c|^2 - \frac{3}{2})^2 + 8\operatorname{Re} c &= (4 - \frac{3}{2})^2 - 8 \cdot \frac{1}{2} \\ &= \frac{9}{4} < 3. \end{aligned}$$

Hence  $P_c$  has an attracting fixed point, by Theorem 4.4(a), and so  $c = -\frac{1}{2} + \frac{1}{2}i$  lies in  $M$ , by Theorem 4.3.

### Problem 4.4

Prove that each of the following points lies in  $M$ .

- (a)  $c = -0.9 + 0.1i$       (b)  $c = 0.2 + 0.5i$

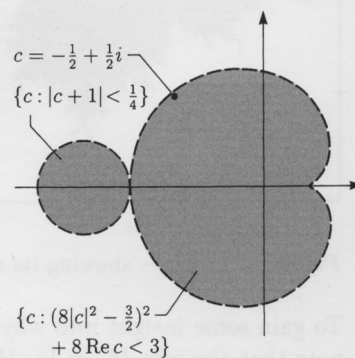


Figure 4.5

An alternative approach is to show that  $|P'_c(\alpha)| < 1$  for one of the fixed points  $\alpha$  of  $P_c$ , but it is messier.



## Proof of Theorem 4.4(a)

First note that  $\alpha$  is a fixed point of  $P_c$  if and only if

$$P_c(\alpha) = \alpha^2 + c = \alpha,$$

that is, if and only if

$$c = \alpha - \alpha^2.$$

Moreover, this fixed point is attracting if and only if

$$|P'_c(\alpha)| = |2\alpha| < 1.$$

Thus  $P_c$  has an attracting fixed point if and only if  $c$  is of the form  $\alpha - \alpha^2$ , where  $|\alpha| < \frac{1}{2}$ , that is, if and only if  $c$  lies in the image of the open disc  $\{z : |z| < \frac{1}{2}\}$  under the function  $z \mapsto z - z^2$ . To understand the nature of this image we use the approach of *Unit D1*, Section 4, expressing the function  $z \mapsto z - z^2$  as a composition of one-one conformal mappings. In fact,

$$\begin{aligned} z - z^2 &= \frac{1}{4} - \left(\frac{1}{4} - z + z^2\right) \\ &= \frac{1}{4} - \left(\frac{1}{2} - z\right)^2, \end{aligned}$$

and so the function  $z \mapsto z - z^2$  has the effect indicated in Figure 4.6.

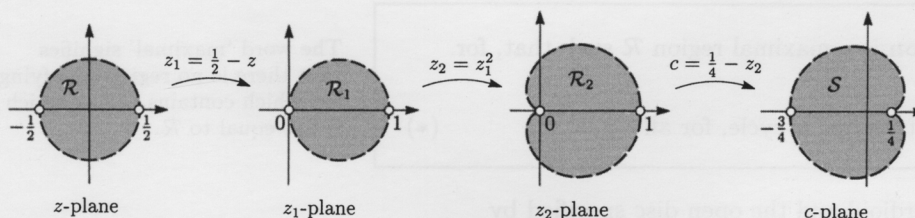


Figure 4.6

Note that the cusp of the cardioid arises because the function  $z \mapsto z - z^2$  is two-one near the point  $\frac{1}{2}$  (see *Unit C2*, Problem 3.3). In particular, this function doubles angles between smooth paths emerging from  $\frac{1}{2}$ .

To complete the proof of part (a), we need to show that  $c$  lies in the region  $S$ , bounded by the cardioid if and only if

$$\left(8|c|^2 - \frac{3}{2}\right)^2 + 8\operatorname{Re} c < 3.$$

This is a rather fiddly technical exercise, which can be done in various ways. One approach is to note that the cardioid itself is the image of the circle  $|z| = \frac{1}{2}$  under the function  $z \mapsto z - z^2$ , that is, it is the path in the  $c$ -plane with parametrization

$$\begin{aligned} \gamma(t) &= \frac{1}{2}e^{it} - \left(\frac{1}{2}e^{it}\right)^2 \\ &= \frac{1}{2}(\cos t + i \sin t) - \frac{1}{4}(\cos 2t + i \sin 2t) \quad (t \in [-\pi, \pi]). \end{aligned}$$

In *Unit A2*, Exercise 2.6, we found that this path has equation

$$4|c|^4 - \frac{3}{2}|c|^2 + \frac{1}{2}\operatorname{Re} c = \frac{3}{64},$$

which can be rearranged to give

$$\left(8|c|^2 - \frac{3}{2}\right)^2 + 8\operatorname{Re} c = 3.$$

It can then be shown that, for  $\frac{1}{4} < r < \frac{3}{4}$ , the cardioid splits the part of the circle  $|c| = r$  in the open upper half-plane into two arcs (see Figure 4.7). On the left arc, which is inside the cardioid, Inequality (4.5) holds, whereas on the right arc we have the opposite inequality. For other positive values of  $r$ , the cardioid does not meet this semi-circle, and the direction of the inequality can be found by considering its nature at  $c = \pm r$ . We omit the details. ■

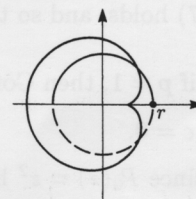


Figure 4.7

### Problem 4.5

(a) Prove that

$$P_c^2(z) - z = (P_c(z) - z)(z^2 + z + c + 1).$$

(b) Deduce from part (a) that if  $c \neq -\frac{3}{4}$ , then  $P_c$  has the 2-cycle  $\alpha_1, \alpha_2$  where

$$\alpha_1 = -\frac{1}{2} + \sqrt{-\frac{3}{4} - c}, \quad \alpha_2 = -\frac{1}{2} - \sqrt{-\frac{3}{4} - c},$$

with multiplier

$$(P_c^2)'(\alpha_1) = 4\alpha_1\alpha_2.$$

What happens if  $c = -\frac{3}{4}$ ?

(c) Deduce from part (b) that  $P_c$  has an attracting 2-cycle if and only if  $|c + 1| < \frac{1}{4}$ .

This proves Theorem 4.4(b).

Theorem 4.4 suggests that each atom in the Mandelbrot set has an associated period  $p$  such that, for each  $c$  in the interior of the atom, the function  $P_c$  has an attracting  $p$ -cycle. Numerical experiments appear to confirm this and so we introduce the notion of a *periodic region*.

**Definition** A **periodic region** is a maximal region  $\mathcal{R}$  such that, for some positive integer  $p$ ,

the function  $P_c$  has an attracting  $p$ -cycle, for all  $c \in \mathcal{R}$ . (\*)

The word 'maximal' signifies that there is no region satisfying (\*) which contains  $\mathcal{R}$  but which is not equal to  $\mathcal{R}$ .

For example, the inside of the cardioid and the open disc specified by Inequalities (4.5) and (4.6) are both periodic regions. Unfortunately, none of the other periodic regions in  $M$  seems to have such a straightforward characterization, but we can obtain some information about their location by using the following result.

**Theorem 4.5** The function  $P_c$  has a super-attracting  $p$ -cycle if and only if

$$P_c^p(0) = 0, \quad \text{but } P_c^k(0) \neq 0, \quad \text{for } k = 1, 2, \dots, p-1. \quad (4.7)$$

**Proof** By Theorem 2.4, any  $p$ -cycle of  $P_c$

$$\alpha, P_c(\alpha), \dots, P_c^{p-1}(\alpha),$$

has multiplier

$$P_c'(\alpha)P_c'(P_c(\alpha)) \cdots P_c'(P_c^{p-1}(\alpha)) = (2\alpha)(2P_c(\alpha)) \cdots (2P_c^{p-1}(\alpha)),$$

since  $P_c'(z) = 2z$ . Therefore, such a  $p$ -cycle is super-attracting if and only if one of the points of the  $p$ -cycle is 0. But  $P_c$  has a  $p$ -cycle including 0 if and only if Condition (4.7) holds, and so the proof is complete. ■

For example, if  $p = 1$ , then Condition (4.7) becomes

$$P_c(0) = c = 0,$$

as expected, since  $P_0(z) = z^2$  has the super-attracting fixed point 0.

If  $p = 2$ , then Condition (4.7) becomes

$$P_c^2(0) = c^2 + c = 0, \quad \text{but } P_c(0) = c \neq 0.$$

The only solution is  $c = -1$ , as expected, since  $P_{-1}(z) = z^2 - 1$  has the super-attracting 2-cycle 0, -1.



### Problem 4.6

Show that the function  $P_c$  has a super-attracting 3-cycle for precisely three different points  $c$ , one of which lies in the interval  $[-1.8, -1.7]$  and the other two of which form a pair of complex conjugates.

It can be proved that if  $P_c$  has an attracting  $p$ -cycle, then  $c$  lies in a periodic region which contains exactly one point  $c_0$ , say, with a super-attracting  $p$ -cycle. We call  $c_0$  the **centre** of the associated periodic region. Figure 4.8 shows the approximate location in the Mandelbrot set of all points  $c_0$  for which  $P_{c_0}$  has a super-attracting  $p$ -cycle, for  $p = 1, 2, 3, 4$ .

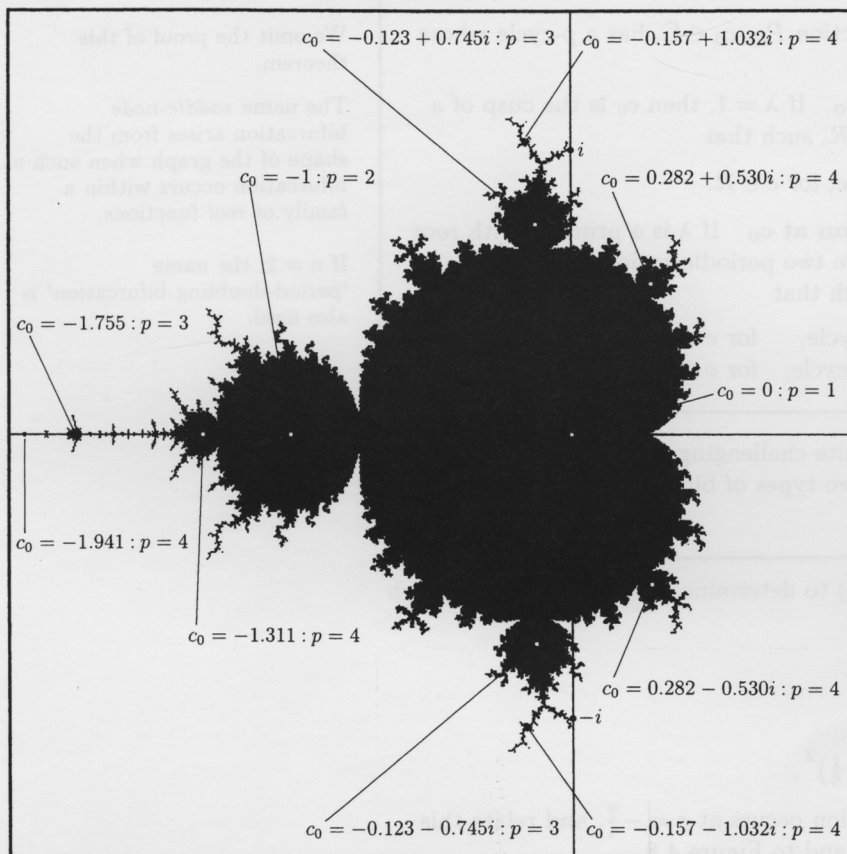


Figure 4.8

In fact the boundaries of all the periodic regions are either (roughly) circular in shape or are shaped like a cardioid but they are linked together in a very complicated manner. We cannot hope to give a full explanation of the way in which the periodic regions of  $M$  fit together, but we can gain some insight into their structure by looking more closely at the two sets given by Theorem 4.4. These are plotted in Figure 4.9.

The key points to consider here are  $c = \frac{1}{4}$ , which is the cusp of the cardioid, and  $c = -\frac{3}{4}$ , where the cardioid and the circle meet. These are the points where the cycle structure of  $P_c$  changes; for example, as  $c$  passes through  $-\frac{3}{4}$  from the cardioid into the disc, the attracting fixed point of  $P_c$  becomes repelling and the repelling 2-cycle of  $P_c$  becomes attracting. At such points, we say that a *bifurcation* occurs. In order to characterize such bifurcations, we look more closely at the cycle structure of  $P_c$  for  $c = \frac{1}{4}$  and  $c = -\frac{3}{4}$ .

For  $c = \frac{1}{4}$ , the function  $P_c = P_{1/4}$  has just one fixed point,  $\alpha = \frac{1}{2}$ , with multiplier

$$P'_c(\alpha) = P'_{1/4}\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) = 1.$$

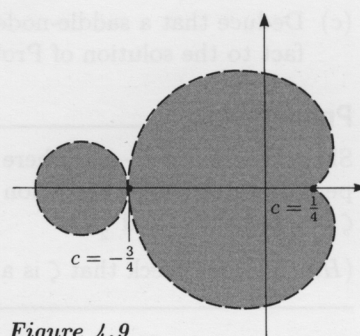


Figure 4.9

For  $c = -\frac{3}{4}$ , the function  $P_c = P_{-3/4}$  has two fixed points, one of which is at  $\alpha = -\frac{1}{2}$ , with multiplier

$$P'_c(\alpha) = P'_{-3/4}(-\frac{1}{2}) = 2(-\frac{1}{2}) = -1.$$

Thus, for both these key points  $c$ , the function  $P_c$  has a fixed point  $\alpha$  whose multiplier  $P'_c(\alpha)$  is a root of unity. In order to state a general result, we say that the number  $\lambda$  is a **primitive  $n$ th root of unity** if  $\lambda$  is a root of unity and if  $n$  is the smallest positive integer for which  $\lambda^n = 1$ . For example,  $-1$  is a primitive square root of unity, but it is not a *primitive* fourth root of unity.

The following description gives the two types of bifurcation which occur when the multiplier of a cycle is a root of unity.

$-1$  is the multiplier we found above for  $c = -\frac{3}{4}$  and  $\alpha = -\frac{1}{2}$ .

**Theorem 4.6** Suppose that the function  $P_{c_0}, c_0 \in \mathbb{C}$ , has a  $p$ -cycle whose multiplier  $\lambda$  is a root of unity.

- (a) **Saddle-node bifurcation at  $c_0$**  If  $\lambda = 1$ , then  $c_0$  is the cusp of a cardioid-shaped periodic region  $\mathcal{R}$ , such that

$P_c$  has an attracting  $p$ -cycle, for  $c \in \mathcal{R}$ .

- (b) **Period-multiplying bifurcation at  $c_0$**  If  $\lambda$  is a primitive  $n$ th root of unity, for  $n > 1$ , then there are two periodic regions  $\mathcal{R}_1$  and  $\mathcal{R}_2$  whose boundaries meet at  $c_0$  such that

$P_c$  has an attracting  $\begin{cases} p\text{-cycle,} & \text{for } c \in \mathcal{R}_1, \\ np\text{-cycle,} & \text{for } c \in \mathcal{R}_2. \end{cases}$

We omit the proof of this theorem.

The name *saddle-node* bifurcation arises from the shape of the graph when such a bifurcation occurs within a family of *real* functions.

If  $n = 2$ , the name 'period-doubling bifurcation' is also used.

In the next two problems, which are quite challenging, we ask you to investigate specific examples of these two types of bifurcations.

#### Problem 4.7

- (a) Use your solution to Problem 2.2(a) to determine a polynomial  $Q_c(z)$  such that

$$P_c^3(z) - z = (P_c(z) - z)Q_c(z).$$

- (b) Verify that

$$Q_{-7/4}(z) = (z^3 + \frac{1}{2}z^2 - \frac{9}{4}z - \frac{1}{8})^2.$$

- (c) Deduce that a saddle-node bifurcation occurs at  $c = -\frac{7}{4}$ , and relate this fact to the solution of Problem 4.6 and to Figure 4.8.

#### Problem 4.8

Show that if  $c = \zeta - \zeta^2$ , where  $2\zeta$  is a root of unity ( $\neq 1$ ), then a period-multiplying bifurcation occurs at  $c$ . Relate this fact to Figure 4.8, with  $\zeta = -\frac{1}{2}, \frac{1}{2}e^{2\pi i/3}$  and  $\frac{1}{2}i$ .

(Hint: First check that  $\zeta$  is a fixed point of  $P_c$ .)

We are now in a better position to describe the structure of the Mandelbrot set  $M$ . Using the approach of Problem 4.8, we find that each part of the main cardioid in Figure 4.8 is decorated by periodic regions. In a similar way all these periodic regions are themselves decorated everywhere by further periodic regions, and so on.

All these period-multiplying bifurcations go a long way towards explaining the complicated structure of  $M$ . In addition, however, we find throughout the boundary of  $M$  the appearance of small cardioid-shaped periodic regions, arising from saddle-node bifurcations, such as the one at  $c = -\frac{7}{4}$  in Problem 4.7. All these cardioid-shaped regions are themselves decorated with smaller periodic regions, as a result of period-multiplying bifurcations, and so

The rest of this subsection is intended for reading only.



they give rise to small copies of the Mandelbrot set within itself (see Figure 4.3(c)). These copies appear to be linked together in  $M$  (remember that the Mandelbrot set is connected) by a complicated network of 'veins'. The simplest such vein lies along the real axis from  $-2$  to  $\frac{1}{4}$ . In fact, one of the unsolved problems about the Mandelbrot set  $M$  is to decide whether  $M$  is not only connected, but also *pathwise* connected.

A pathwise connected set must be connected, but a connected set need not be pathwise connected. However, for open sets the two types of connectedness are equivalent.

## The real bifurcation diagram

Finally, we describe a very effective graphical method of obtaining information about the periodic regions of the Mandelbrot set whose centres lie on the real axis. The idea is to choose a large number of real values of  $c$  between  $-2$  and  $\frac{1}{4}$ , and plot each of the corresponding sequences  $\{P_c^n(0)\}$  vertically above and below a horizontal  $c$ -axis (see Figure 4.10(a)). In order to determine any attracting  $p$ -cycles (to which this sequence will be attracted, by the remark following Theorem 4.3) the first 200 or so terms are discarded and the next 600 or so are plotted. Thus if  $P_c$  has an attracting  $p$ -cycle, then  $p$  points should be plotted above and below the corresponding point  $c$ .

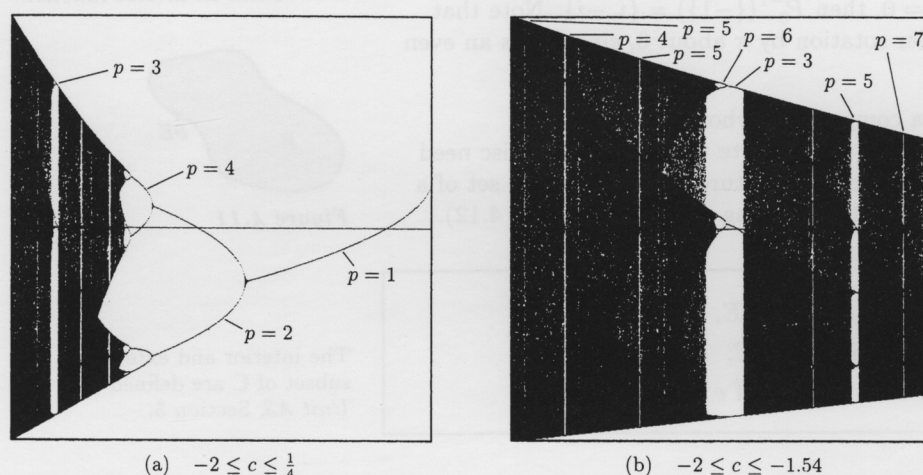


Figure 4.10

As expected, this 'bifurcation diagram' reveals the convergence of  $\{P_c^n(0)\}$  to an attracting fixed point for  $-\frac{3}{4} < c < \frac{1}{4}$ , to an attracting 2-cycle for  $-\frac{5}{4} < c < -\frac{3}{4}$ , to an attracting 4-cycle for  $c$  just to the left of  $-\frac{5}{4}$ , with further period-doubling bifurcations to the left of this. Also visible is an attracting 3-cycle for  $c$  just to the left of  $-\frac{7}{4}$ . For other values of  $c$ , it is less clear what is happening, but by scaling the  $c$ -axis appropriately (see Figure 4.10(b)) many other 'periodic windows' are revealed which correspond to attracting  $p$ -cycles, some of which are labelled in the diagram. Notice that in each such window there is a point  $c$  whose cycle includes 0; this value of  $c$  is the centre of the corresponding periodic region in the Mandelbrot set. In fact, pictures of this type for the related family of iteration sequences given by

$$x_{n+1} = kx_n(1 - x_n), \text{ with } x_0 = \frac{1}{2}, \text{ where } 0 \leq k \leq 4,$$

were studied in the early 1970s, before the Mandelbrot set itself had been plotted. In particular, the order in which the periodic windows appear was found and the rate at which period-doubling occurs was discovered (by M. Feigenbaum) to have a certain universal property. Thus, even the part of the Mandelbrot set which lies on the real axis is extremely complicated, and there are still unanswered questions about it. For example, it is believed that every non-empty open interval of  $[-2, 1/4]$  meets at least one of the periodic windows, but at the time of writing (1993) this has still not been proved.

### 4.3 Outline proofs of Theorems 4.1 and 4.3

The aim of this subsection is to indicate, without going into all the details, why Theorems 4.1 and 4.3 are true.

This subsection may be omitted on a first reading.

#### Theorem 4.1 Fatou–Julia Theorem

For any  $c \in \mathbb{C}$ ,

$$K_c \text{ is connected} \iff 0 \in K_c.$$

#### Theorem 4.3 If the function $P_c$ has an attracting cycle, then $c \in M$ .

We shall need the concept of the **preimage set**  $P_c^{-1}(E)$  of a set  $E$  under the function  $P_c$ . This is simply the set of points which are mapped to  $E$  by  $P_c$ :

$$P_c^{-1}(E) = \{z : P_c(z) \in E\}.$$

For example, if  $E = \{-1\}$  and  $c = 0$ , then  $P_0^{-1}(\{-1\}) = \{i, -i\}$ . Note that  $P_c^{-1}(E)$  is always symmetric under rotation by  $\pi$  about 0, since  $P_c$  is an even function.

We say that a **compact disc** is a compact set whose boundary is a simple-closed smooth path (see Figure 4.11). Note that a compact disc need not be a disc, but it must be connected. The nature of the preimage set of a compact disc  $E$  under  $P_c$  is given by the following result (see Figure 4.12).

This notation does *not* imply that  $P_c$  has an inverse function.

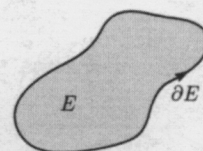
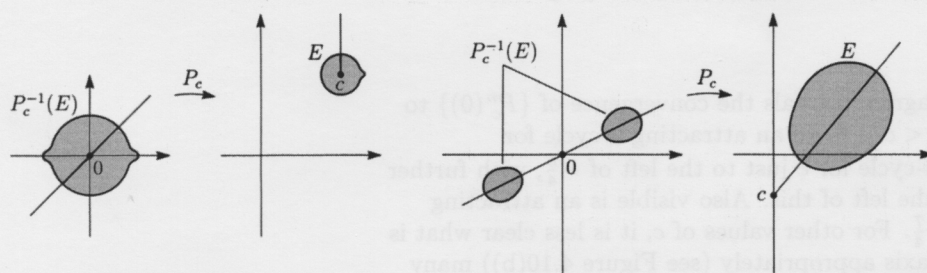


Figure 4.11

**Lemma 4.1** If  $E$  is a compact disc and  $c \notin \partial E$ , then  $P_c^{-1}(E)$  is

- (a) one compact disc containing 0, if  $c \in \text{int } E$ ;
- (b) two compact discs, neither containing 0, if  $c \in \text{ext } E$ .

The interior and exterior of a subset of  $\mathbb{C}$  are defined in Unit A3, Section 5.



If  $c \in \partial E$ , then  $P_c^{-1}(E)$  forms a filled-in figure of eight.

Figure 4.12

To indicate why Lemma 4.1 is true we have shown in both parts of Figure 4.12 a typical ray emerging from the point  $c$  and meeting the set  $E$ , as well as the preimage of this ray which consists of the two rays emerging from 0, combining to form a line through 0.

Now we start the outline proof of Theorem 4.1 by choosing a closed disc

$$E_0 = \{z : |z| \leq r\}, \text{ where}$$

$$r > \max\{r_c, |c|\}$$

and

$$P_c^n(0) \notin \partial E_0, \text{ for } n = 1, 2, \dots$$

$$(4.8) \quad r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$$

$$(4.9)$$

Then we define the sequence of successive preimage sets of  $E_0$  under  $P_c$ :

$$E_1 = P_c^{-1}(E_0), \quad E_2 = P_c^{-1}(E_1), \quad \text{and so on,}$$

Note that the sets  $E_0, E_1, E_2, \dots$  are not escape sets here.



that is, we put

$$E_n = \{z : P_c^n(z) \in E_0\}, \quad \text{for } n = 1, 2, \dots$$

If  $P_c^{n+1}(z) \in E_0$ , then we must have  $P_c^n(z) \in E_0$  also (by Theorem 2.2, because  $r > r_c$ ). Hence

$$E_{n+1} \subseteq E_n, \quad \text{for } n = 0, 1, 2, \dots,$$

so that the sets  $E_n$  are nested. Moreover,

$$\begin{aligned} E_0 \cap E_1 \cap E_2 \cap \dots &= \{z : P_c^n(z) \in E_0, \text{ for } n = 0, 1, 2, \dots\} \\ &= \{z : |P_c^n(z)| \leq r, \text{ for } n = 0, 1, 2, \dots\} \\ &= K_c, \end{aligned} \quad (4.10)$$

by Theorem 2.2. Thus the shape of  $K_c$  is determined by the shapes of the sets  $E_n$ .

Figure 4.13(a) and (b) illustrate the first few of these nested sets  $E_n$ ,  $n = 1, 2, \dots$ , for the two cases  $c = -1$  and  $c = 1$ . In both cases we have taken  $E_0 = \{z : |z| \leq r\}$ , where  $r = 1.8$ , so that Conditions (4.8) and (4.9) are satisfied in each case. Points of  $E_n - E_{n+1}$  are plotted black if  $n$  is even and white if  $n$  is odd.

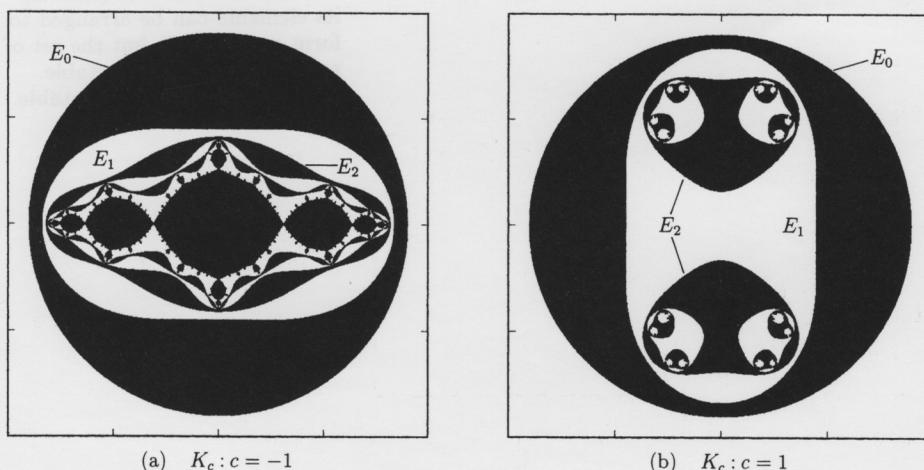


Figure 4.13

In Figure 4.13(a), all the sets  $E_n$  are compact discs. However, in Figure 4.13(b) the sets  $E_0$  and  $E_1$  are compact discs, but  $E_2$  consists of two compact discs,  $E_3$  consists of four compact discs, and so on. We now show that these different structures are related to whether or not the point 0 (and hence  $c$ ) lies in  $K_c$ .

First assume that  $0 \in K_c$ , so that  $c = P_c(0) \in K_c$ . Then

$$c \in \text{int } E_n, \quad \text{for } n = 0, 1, 2, \dots,$$

so that, by Lemma 4.1(a),

$$E_{n+1} = P_c^{-1}(E_n) \text{ is one compact disc, for } n = 0, 1, 2, \dots$$

Thus  $E_n$ ,  $n = 0, 1, 2, \dots$ , is a nested sequence of connected compact sets (as in Figure 4.13(a)), and it follows from this (though we do not give the details) that  $K_c = E_0 \cap E_1 \cap \dots$  is also connected, as required.

Next assume that  $0 \notin K_c$ , so that  $c = P_c(0) \notin K_c$ . Since  $c \in \text{int } E_0$ , there is a positive integer  $m$  such that

$$c \in \text{int } E_{m-1}, \quad \text{but } c \notin E_m.$$

Thus, by Lemma 4.1(b),  $E_m$  is one compact disc, but  $E_{m+1}$  is two; see Figure 4.14 (which is based on Figure 4.13(b)), in which  $c = 1$  and  $m = 1$ .

Note that

$c \notin \partial E_n$ , for  $n = 0, 1, 2, \dots$ ,  
since  $P_c^n(c) = P_c^{n+1}(0) \notin \partial E_0$ ,  
for  $n = 0, 1, 2, \dots$ , by  
Condition (4.9).

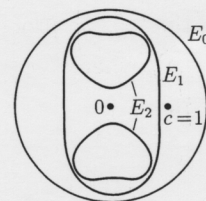


Figure 4.14

Thus, the set  $K_c$  lies in the union of the two halves of  $E_{m+1}$ , and it has points in each half because both  $K_c$  and  $E_{m+1}$  are symmetric under rotation by  $\pi$  about 0. Hence  $K_c$  is disconnected, as required. This completes the outline proof of Theorem 4.1.

Notice from Figure 4.14 that  $c$  lies outside both halves of  $E_{m+1}$  and so, by Lemma 4.1(b), the set  $E_{m+2} = P_c^{-1}(E_{m+1})$  consists of four compact discs (two for each half of  $E_{m+1}$ ). In general,  $E_{m+n}$  consists of  $2^n$  compact discs. Now,  $K_c$  is a subset of each of these preimage sets  $E_{m+n}$ , and so any connected subset  $\tilde{K}_c$  of  $K_c$  must lie in exactly one of the  $2^n$  compact discs comprising  $E_{m+n}$ , say  $\tilde{E}_{m+n}$ . Thus

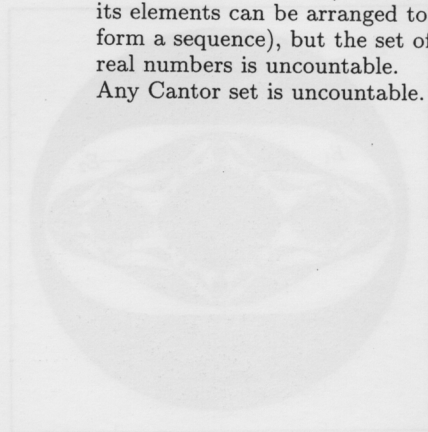
$$\tilde{E}_{m+1} \supseteq \tilde{E}_{m+2} \supseteq \cdots \supseteq \tilde{K}_c.$$

It can be proved that the diameters of these compact discs  $\tilde{E}_{m+n}$  tend to zero as  $n \rightarrow \infty$ , so that  $\tilde{K}_c$  must be a singleton set, and there will be one such singleton set for each of the (infinitely many) possible nested sequences of compact discs  $\tilde{E}_{m+n}$ . A set obtained by a construction of this type is called a **Cantor set**, and so  $K_c$  is such a Cantor set for each  $c$  not in  $M$ .

It follows that if  $c \notin M$ , then  $K_c$  has no interior points and so the function  $P_c$  cannot have an attracting cycle, in view of Theorem 2.5(a). This shows one possible way to prove Theorem 4.3.

This requires a rather tricky application of the Riemann Mapping Theorem and Schwarz's Lemma.

Georg Cantor (1845–1918) developed the foundations of 'set theory'. For example, he showed that the set of rational numbers is countable (that is, its elements can be arranged to form a sequence), but the set of real numbers is uncountable. Any Cantor set is uncountable.





## 5 BEYOND THE MANDELBROT SET

After reading through this section, you should be able to:

- (a) appreciate the universal nature of the Mandelbrot set.

This section is intended for reading only.

In Subsection 1.4 we discussed briefly the Newton–Raphson function

$$N(z) = \frac{2z^3 + 1}{3z^2}, \quad (5.1)$$

corresponding to the cubic polynomial function  $p(z) = z^3 - 1$ .

Under iteration of  $N$ , all points of  $\mathbb{C}$  are attracted to one of the zeros of  $p$ , or else they remain on the common basin boundary (see Figure 1.9), which includes the point at  $\infty$  for the extended function  $\hat{N}$ .

Following the discovery of the Mandelbrot set, the Newton–Raphson method for a general cubic function was investigated by computer, in the early 1980s. For most cubic functions, the corresponding Newton–Raphson function behaves under iteration in the same way as the function in Equation (5.1), but for some cubic functions a difference was found.

To make this difference precise, we consider the family of cubic functions given by

$$\begin{aligned} p_c(z) &= (z - 1)\left(z + \frac{1}{2} - c\right)\left(z + \frac{1}{2} + c\right) \\ &= z^3 - \left(\frac{3}{4} + c^2\right)z - \frac{1}{4} + c^2, \end{aligned} \quad (5.2)$$

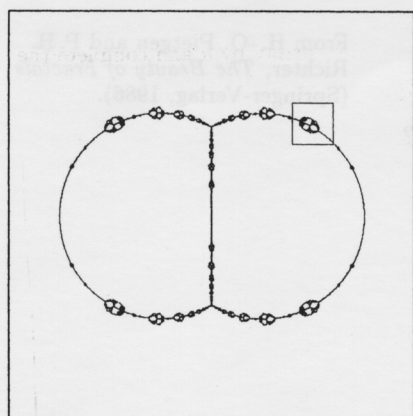
For example, if  $c = \pm(\sqrt{3}/2)i$ , then

$$p_c(z) = z^3 - 1.$$

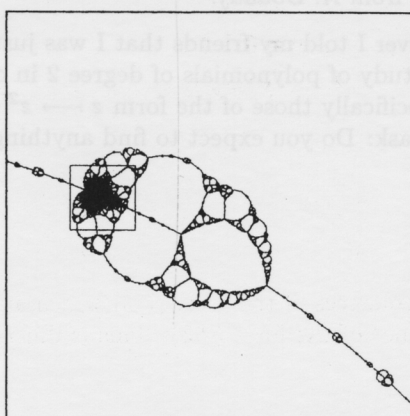
where  $c \in \mathbb{C}$ . The corresponding Newton–Raphson function is

$$N_c(z) = z - \frac{p_c(z)}{p'_c(z)} = \frac{2z^3 + \left(\frac{1}{4} - c^2\right)}{3z^2 - \left(\frac{3}{4} + c^2\right)},$$

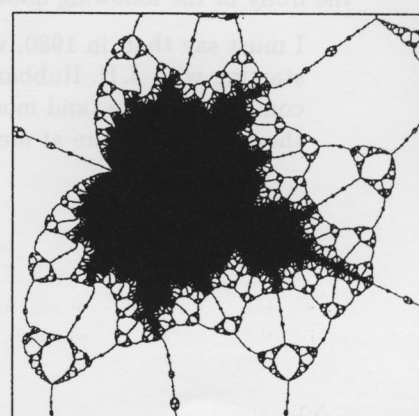
and it is a straightforward matter to check that the critical points of  $N_c$  (that is, the points where  $N'_c$  vanishes) are the three zeros of  $p_c$  and the point 0. Usually, the critical point 0 is attracted to one of the zeros of  $p_c$  under iteration of  $N_c$ , although it may also remain on the basin boundary (for example, if  $c = \pm(\sqrt{3}/2)i$ , then  $\hat{N}_c(0) = \infty$  and  $\hat{N}_c(\infty) = \infty$ ). For some values of  $c$ , however, the function  $N_c$  has an attracting  $p$ -cycle, where  $p > 1$ , to which the point 0 is attracted. In Figure 5.1(a), we have plotted in the parameter plane those values of  $c$  for which the sequence  $\{N_c^n(0)\}$  does not converge to one of the zeros of  $p_c$ .



(a)



(b)



(c)

Figure 5.1  $\{c : N_c^n(0) \nrightarrow \text{a zero of } p_c\}$

If we zoom in on various parts of this set, then we find small copies of the Mandelbrot set! Inspired by examples of this type, Douady and Hubbard developed a theory which shows that copies of the Mandelbrot set appear in the parameter plane whenever we consider the iteration of suitable families of analytic functions. Thus the Mandelbrot set has a universal nature!

The boxes represented in Figures 5.1(a) and 5.2(a) are each

$$\{c : -2 \leq \operatorname{Re} c, \operatorname{Im} c \leq 2\}.$$

To emphasize this universal nature, we describe one relation of the Mandelbrot set, which is obtained by iterating *non-analytic* functions of the form

$$f_c(z) = \bar{z}^2 + c,$$

where  $c \in \mathbb{C}$ . By analogy with Theorem 4.1, Corollary 1, we plot the set

$$\{c : |f_c^n(0)| \leq 2, \text{ for } n = 0, 1, 2, \dots\},$$

which is called the **tricorn**, or **Mandelbar set**; see Figure 5.2(a).

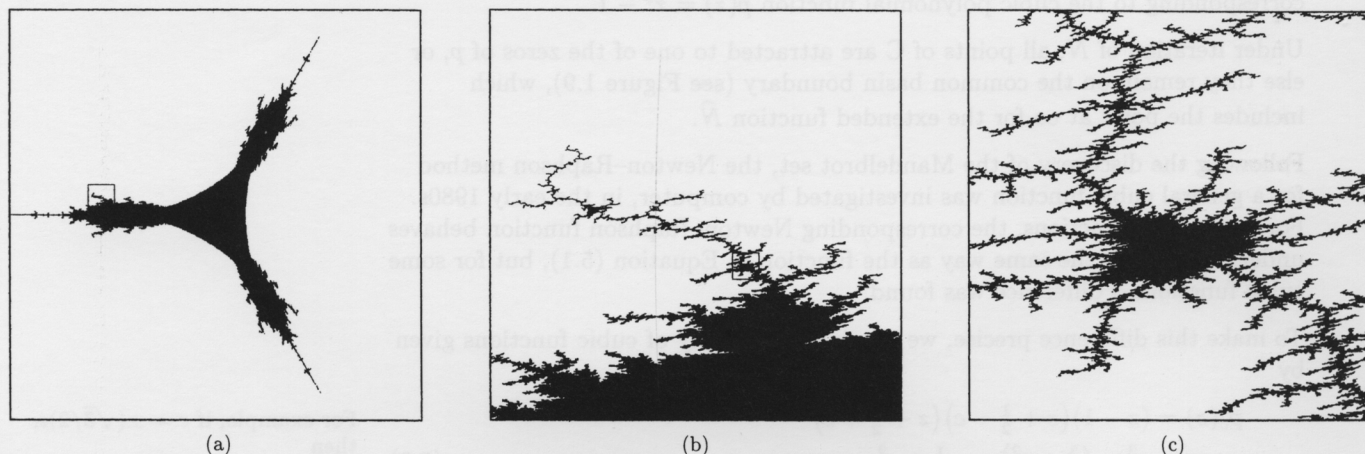


Figure 5.2 The tricorn

This set is symmetric under rotation by  $2\pi/3$  about 0 (as well as under reflection in the real axis), and it appears to be connected. Closer inspection reveals that some parts of the boundary of the tricorn are smooth (for example near  $c = \frac{1}{4}$ ), whereas other parts are extremely irregular (see Figure 5.2(b)). As you might expect, the tricorn contains small copies of itself, but it also contains small copies of the Mandelbrot set — a truly universal object!

In this unit we have only scratched the surface of the subject of complex iteration. For example, there is much more to be said about the structure of the individual Julia sets  $J_c$ , and there are many families of entire functions (such as  $z \mapsto e^{cz}$ , where  $c$  is a complex parameter), whose behaviour under iteration leads to completely new phenomena. Nevertheless, we hope that you have gained some insight into this remarkable subject, and that you appreciate the irony in the following quotation from A. Douady.

I must say that, in 1980, whenever I told my friends that I was just starting with J. H. Hubbard a study of polynomials of degree 2 in one complex variable (and more specifically those of the form  $z \mapsto z^2 + c$ ), they would all stare at me and ask: Do you expect to find anything new?

From H.-O. Pietgen and P. H. Richter, *The Beauty of Fractals* (Springer-Verlag, 1986).



# EXERCISES

## Section 1

**Exercise 1.1** Plot the terms  $z_0, z_1, z_2, z_3$  for each of the following iteration sequences, and write down the corresponding function  $f$ .

- (a)  $z_{n+1} = iz_n, \quad z_0 = 2i$   
 (b)  $z_{n+1} = 2z_n(1 - z_n), \quad z_0 = \frac{1}{4}$   
 (c)  $z_{n+1} = \frac{z_n}{z_n + 1}, \quad z_0 = \frac{1}{2}(-1 + i)$

**Exercise 1.2** Find all the fixed points of each of the following functions and classify them as (super-)attracting, repelling or indifferent.

- (a)  $f(z) = z - z^2$  (b)  $f(z) = 2z(1 - z)$   
 (c)  $f(z) = z^2 - \frac{1}{2}$  (d)  $f(z) = z/(z + 1)$

**Exercise 1.3** Prove that any iteration sequence of the form

$$z_{n+1} = 2z_n(1 - z_n), \quad n = 0, 1, 2, \dots,$$

is conjugate, via the conjugating function  $h(z) = 1 - 2z$ , to one of the form

$$w_{n+1} = w_n^2, \quad n = 0, 1, 2, \dots$$

Hence obtain a formula for  $z_n$  in terms of  $z_0$ .

**Exercise 1.4** Let  $f(z) = (az + b)/(cz + d)$ , where  $ad - bc \neq 0$  and  $c \neq 0$ , let  $\hat{f}$  be the corresponding extended Möbius transformation, and let  $z_n = \hat{f}^n(z_0)$ , where  $z_0 \in \hat{\mathbb{C}}$ .

This exercise, which is fairly challenging, contains a complete analysis of the behaviour of iteration sequences defined by Möbius transformations.

- (a) Prove that if  $(a - d)^2 + 4bc = 0$ , then  $\hat{f}$  has a unique fixed point  $\alpha$ , which lies in  $\mathbb{C}$ . Deduce that the sequence  $\{z_n\}$  is conjugate, via the conjugating function  $h(z) = 1/(z - \alpha)$ , to an iteration sequence of the form

$$w_{n+1} = w_n + 2c/(a + d), \quad n = 0, 1, 2, \dots$$

Hence prove that  $z_n \rightarrow \alpha$  as  $n \rightarrow \infty$ , for all  $z_0 \in \hat{\mathbb{C}}$ .

- (b) Prove that if  $(a - d)^2 + 4bc \neq 0$ , then  $\hat{f}$  has two fixed points  $\alpha$  and  $\beta$ , both lying in  $\mathbb{C}$ . Deduce that  $\{z_n\}$  is conjugate, via the conjugating function  $h(z) = (z - \alpha)/(z - \beta)$ , to an iteration sequence of the form

$$w_{n+1} = f'(\alpha)w_n, \quad n = 0, 1, 2, \dots$$

Hence prove that if  $|f'(\alpha)| < 1$ , then  $z_n \rightarrow \alpha$  as  $n \rightarrow \infty$ , for all

$z_0 \in \hat{\mathbb{C}} - \{\beta\}$ . Also describe the behaviour of  $\{z_n\}$  if

- (i)  $|f'(\alpha)| > 1$ , (ii)  $|f'(\alpha)| = 1$ .

(Hint: You will need to use the fact that  $\alpha$  and  $\beta$  are fixed points of  $f$  to simplify the algebra.)

## Section 2

**Exercise 2.1** Use Theorem 2.1 to show that the iteration sequence

$$z_{n+1} = 3z_n(1 - z_n), \quad n = 0, 1, 2, \dots,$$

where  $z_0 = \frac{1}{2}$ , is conjugate to an iteration sequence of the form

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \dots$$

**Exercise 2.2**

(a) Prove that if  $|c| \leq \frac{1}{4}$ , then

$$|z| \leq \frac{1}{2} + \sqrt{\frac{1}{4} - |c|} \implies |P_c(z)| \leq \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}.$$

(b) Deduce from part (a) that if  $|c| \leq \frac{1}{4}$ , then

$$\left\{ z : |z| \leq \frac{1}{2} + \sqrt{\frac{1}{4} - |c|} \right\} \subseteq K_c.$$

(c) Combine the result of part (b) with Theorem 2.3(a) to show that, when  $c$  is close to 0, the set  $K_c$  is approximately equal to the closed unit disc.

**Exercise 2.3** For each of the following functions  $f$  and points  $\alpha$ , show that  $\alpha$  is a periodic point of  $f$  and decide whether it is (super-)attracting, repelling or indifferent.

(a)  $f(z) = -z, \quad \alpha = i$       (b)  $f(z) = z^2 - 2, \quad \alpha = \frac{1}{2}(-1 + \sqrt{5})$

(c)  $f(z) = z^3 + i, \quad \alpha = 0$       (d)  $f(z) = z^3, \quad \alpha = e^{\pi i/13}$

**Exercise 2.4** This exercise relates to Properties (2.12) and (2.13) for the function  $P_0(z) = z^2$ , whose Julia set is the unit circle.

In order to attempt this exercise you will need to have read Subsection 2.4.

- (a) Determine the repelling periodic points of  $P_0$  and prove that each arc of the unit circle contains such a repelling periodic point.
- (b) Determine the backward iterates under  $P_0$  of the point 1 and prove that each arc of the unit circle contains such a backward iterate.

## Section 3

**Exercise 3.1** Let  $P_{1/4}(x) = x^2 + \frac{1}{4}$ .

Plot  $y = P_{1/4}(x)$  and  $y = x$  on the same diagram and use graphical iteration to plot the iteration sequence

$$x_{n+1} = P_{1/4}(x_n), \quad n = 0, 1, 2, \dots,$$

with  $x_0 = 0$ . Describe the behaviour of the sequence  $\{x_n\}$ .

**Exercise 3.2**

(a) Use graphical iteration to show that any iteration sequence of the form

$$x_{n+1} = \frac{x_n}{x_n + 1}, \quad n = 0, 1, 2, \dots,$$

with  $x_0 \in \mathbb{R} - A$ , where  $A = \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\}$ , converges to the point 0. (Note that if  $x = -1/n$ , then  $x/(x+1) = -1/(n-1)$ .)

(b) Relate your answer to part (a) to Exercise 1.4(a).



## Section 4

**Exercise 4.1** Use Corollary 1 to Theorem 4.1 to determine which of the following points  $c$  lie in  $M$ .

- (a)  $c = -1 + 2i$       (b)  $c = -1$       (c)  $c = -1 + i$

**Exercise 4.2** Show that if  $|c| = 2$  but  $c \neq -2$ , then

$$|c^2 + c| > 2,$$

and deduce that  $c \notin M$ .

(Hint: Use the factorization  $c^2 + c = c(c + 1)$ .)

**Exercise 4.3** Let  $M_n = \{c : |P_c^n(0)| \leq 2\}$ , for  $n = 1, 2, \dots$ .

- (a) Prove that  $M_n \subseteq M_1$ , for  $n = 2, 3, \dots$ .

(Hint: If  $|c| > 2$ , then  $|c| > r_c$ , by Figure 2.2.)

- (b) Prove that  $M_{n+1} \subseteq M_n$ , for  $n = 1, 2, \dots$ .

(Hint: If  $|c| \leq 2$ , then  $2 \geq r_c$ , by Figure 2.2.)

**Exercise 4.4** Prove that each of the following points lies in  $M$ .

- (a)  $c = -1.1 - 0.1i$       (b)  $c = 0.6i$

### Exercise 4.5

- (a) Prove that if  $\mathcal{R}$  is a periodic region of  $M$ , then  $\partial\mathcal{R} \subseteq M$ .

- (b) Deduce from part (a) that each of the following points lies in  $M$ .

- (i)  $c = \frac{1}{4} + \frac{1}{2}i$       (ii)  $c = -1 + \frac{1}{4}i$

(Hint: Locate these points in Figure 4.8)

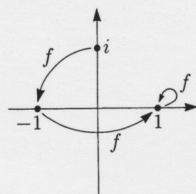
**Exercise 4.6** Explain why there can be at most six values of  $c$  (see Figure 4.8) for which  $P_c$  has a super-attracting 4-cycle.

**Exercise 4.7** Show that a period-doubling bifurcation occurs at  $c = -\frac{5}{4}$ , and relate this fact to Figure 4.8.

# SOLUTIONS TO THE PROBLEMS

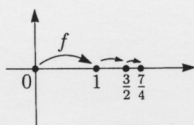
## Section 1

1.1 (a)  $i, -1, 1, 1$ .



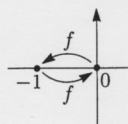
Here  $f(z) = z^2$ .

(b)  $0, 1, \frac{3}{2}, \frac{7}{4}$ .



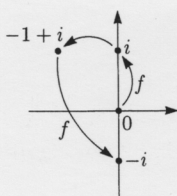
Here  $f(z) = \frac{1}{2}z + 1$ .

(c)  $0, -1, 0, -1$ .



Here  $f(z) = z^2 - 1$ .

(d)  $0, i, -1 + i, -i$ .



Here  $f(z) = z^2 + i$ .

1.2 If  $f(z) = \frac{1}{2}z + 1$ , then

$$\begin{aligned} f^2(z) &= f(f(z)) \\ &= f\left(\frac{1}{2}z + 1\right) \\ &= \frac{1}{2}\left(\frac{1}{2}z + 1\right) + 1 \\ &= \frac{1}{4}z + \frac{3}{2}. \end{aligned}$$

Also

$$\begin{aligned} f^3(z) &= f^2(f(z)) \\ &= \frac{1}{4}\left(\frac{1}{2}z + 1\right) + \frac{3}{2} \\ &= \frac{1}{8}z + \frac{7}{4}. \end{aligned}$$

1.3 (a) If  $f(z) = z + b$ , then

$$\begin{aligned} f^1(z) &= z + b, \\ f^2(z) &= (z + b) + b = z + 2b, \\ f^3(z) &= (z + 2b) + b = z + 3b, \end{aligned}$$

and, in general,

$$f^n(z) = z + nb, \quad \text{for } n = 1, 2, \dots$$

(b) If  $f(z) = z^3$ , then

$$\begin{aligned} f^1(z) &= z^3, \\ f^2(z) &= (z^3)^3 = z^9, \\ f^3(z) &= (z^9)^3 = z^{27}, \end{aligned}$$

and, in general,

$$f^n(z) = z^{3^n}, \quad \text{for } n = 1, 2, \dots$$

1.4 (a) If  $z_0 = 1$ , then we have

$$z_n = 1, \quad \text{for } n = 1, 2, \dots,$$

so that  $\{z_n\}$  is constant, and converges to 1.

(b) If  $z_0 = -i$ , then the terms of the sequence are

$$-i, -1, 1, 1, 1, \dots,$$

so that  $\{z_n\}$  is eventually constant, and converges to 1.

(c) If  $z_0 = e^{2\pi i/3}$ , then the terms of the sequence are

$$e^{2\pi i/3}, e^{4\pi i/3}, e^{8\pi i/3} = e^{2\pi i/3}, \dots,$$

so that the sequence cycles endlessly between these two values.

(d) Since  $z_n = z_0^{2^n}$  and  $|1/z_0| < 1$ , we deduce that

$$\frac{1}{z_n} = \left(\frac{1}{z_0}\right)^{2^n}, \quad n = 1, 2, \dots,$$

is a null sequence. Hence  $z_n \rightarrow \infty$  as  $n \rightarrow \infty$ , by the Reciprocal Rule (Unit A3, Theorem 1.5).

1.5 (a) The fixed point equation is  $f(z) = \frac{1}{2}z + 1 = z$ , and

$$\frac{1}{2}z + 1 = z \iff \frac{1}{2}z = 1,$$

so the only fixed point is 2.

(b) The fixed point equation is  $f(z) = z^2 - 2 = z$ , and

$$z^2 - 2 = z \iff z^2 - z - 2 = 0,$$

so the only fixed points are 2 and -1.

(c) The fixed point equation is  $f(z) = z^3 = z$ , and

$$z^3 = z \iff z(z^2 - 1) = 0,$$

so the only fixed points are 0, 1 and -1.

1.6 (a) Since  $f'(z) = 2z$ , we have

$|f'(0)| = 0 < 1$ , so 0 is an attracting (in fact, a super-attracting) fixed point of  $f$ ;

$|f'(1)| = 2 > 1$ , so 1 is a repelling fixed point of  $f$ .

(b) Since  $f'(z) = \frac{1}{2}$ , we have

$|f'(2)| = \frac{1}{2} < 1$ , so 2 is an attracting fixed point of  $f$ .

(c) Since  $f'(z) = 2z$ , we have

$|f'(2)| = 4 > 1$ , so 2 is a repelling fixed point of  $f$ .



**1.7 (a)** From Example 1.2(a), we know that, for  $n = 1, 2, \dots$ ,

$$f^n(z) = \frac{1}{2^n} z$$

$\rightarrow 0$  as  $n \rightarrow \infty$ , for all  $z \in \mathbb{C}$ .

Hence the basin of attraction of 0 under  $f$  is

$$\{z : f^n(z) \rightarrow 0 \text{ as } n \rightarrow \infty\} = \mathbb{C}.$$

**(b)** From Problem 1.3(b), we know that, for  $n = 1, 2, \dots$ ,

$$f^n(z) = z^{3^n}.$$

Hence

$$f^n(z_0) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for } |z_0| < 1,$$

but

$$f^n(z_0) \not\rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for } |z_0| \geq 1.$$

Thus the basin of attraction of 0 under  $f$  is

$$\{z : f^n(z) \rightarrow 0 \text{ as } n \rightarrow \infty\} = \{z : |z| < 1\}.$$

**1.8** First we note that the function  $h(z) = -z + \frac{1}{2}$  is one-one on  $\mathbb{C}$ . If  $w_n = h(z_n) = -z_n + \frac{1}{2}$ , then  $z_n = -w_n + \frac{1}{2}$ , so

$$z_{n+1} = z_n - z_n^2, \quad n = 0, 1, 2, \dots,$$

becomes

$$\begin{aligned} -w_{n+1} + \frac{1}{2} &= (-w_n + \frac{1}{2}) - (-w_n + \frac{1}{2})^2 \\ &= -w_n + \frac{1}{2} - w_n^2 + w_n - \frac{1}{4}, \end{aligned}$$

for  $n = 0, 1, 2, \dots$ ,

that is,

$$w_{n+1} = w_n^2 + \frac{1}{4}, \quad \text{for } n = 0, 1, 2, \dots,$$

which demonstrates that these two sequences are conjugate.

If  $z_0 = \frac{1}{2}$ , then  $w_0 = 0$ .

**1.9 (a)** First we note that the function

$$h(z) = z + b/(a-1),$$

where  $a \neq 1$ , is one-one on  $\mathbb{C}$ .

If  $w_n = h(z_n) = z_n + b/(a-1)$ , then

$$z_n = w_n - b/(a-1), \text{ so}$$

$$z_{n+1} = az_n + b, \quad n = 0, 1, 2, \dots,$$

becomes

$$w_{n+1} - b/(a-1) = a(w_n - b/(a-1)) + b,$$

for  $n = 0, 1, 2, \dots$ ;

that is,

$$w_{n+1} = aw_n, \quad \text{for } n = 0, 1, 2, \dots, \quad (1)$$

since  $-b/(a-1) = -ab/(a-1) + b$ .

**(b)** The iteration sequence (1) has general term  $w_n = a^n w_0$  (see Example 1.2(a)). Hence

$$\begin{aligned} z_n &= w_n - \frac{b}{a-1} \\ &= a^n w_0 - \frac{b}{a-1}, \quad \text{for } n = 0, 1, 2, \dots, \end{aligned}$$

giving

$$z_n = a^n \left( z_0 + \frac{b}{a-1} \right) - \frac{b}{a-1}, \quad \text{for } n = 0, 1, 2, \dots;$$

that is,

$$z_n = a^n z_0 - \frac{b(1-a^n)}{a-1}, \quad \text{for } n = 0, 1, 2, \dots$$

(i) If  $|a| < 1$ , then  $\{a^n\}$  is a null sequence, so in this case

$$z_n \rightarrow \frac{b}{1-a} \text{ as } n \rightarrow \infty.$$

(Note that  $b/(1-a)$  is the only fixed point of the function  $f(z) = az + b$ .)

(ii) If  $|a| = 1$ ,  $a \neq 1$ , then  $\{a^n\}$  is divergent, by Unit A3, Theorem 1.7(b). It follows that  $\{z_n\}$  is divergent in this case (unless  $z_0 = b/(1-a)$ , in which case  $\{a_n\}$  is constant).

(iii) If  $|a| > 1$ , then  $\{a^n\}$  tends to infinity, by Unit A3, Theorem 1.7(a). It follows that  $\{z_n\}$  tends to infinity in this case (unless  $z_0 = b/(1-a)$ ).

**1.10** With  $N(z) = \frac{z^2 - b}{2z + a}$  and  $h(z) = \frac{z - \alpha}{z - \beta}$ , we have

$$\begin{aligned} h(N(z)) &= \left( \frac{z^2 - b}{2z + a} - \alpha \right) / \left( \frac{z^2 - b}{2z + a} - \beta \right) \\ &= \frac{z^2 - b - \alpha(2z + a)}{z^2 - b - \beta(2z + a)} \\ &= \frac{z^2 - 2\alpha z - (a\alpha + b)}{z^2 - 2\beta z - (a\beta + b)} \\ &= \frac{z^2 - 2\alpha z + \alpha^2}{z^2 - 2\beta z + \beta^2} \quad (\text{by the hint}) \\ &= \frac{(z - \alpha)^2}{(z - \beta)^2} \\ &= (h(z))^2, \end{aligned}$$

as required.

**1.11** If  $p(z) = (z - \alpha)^2$ , then

$$N(z) = z - \frac{(z - \alpha)^2}{2(z - \alpha)} = \frac{1}{2}(z + \alpha),$$

and so, by Problem 1.9(b)(i) (with  $a = \frac{1}{2}$ ,  $b = \frac{1}{2}\alpha$ ), all points of  $\mathbb{C}$  are attracted to  $\alpha$  under iteration of  $N$ .

## Section 2

**2.1 (a)** By Theorem 2.1, with  $a = -4$ ,  $b = 4$ ,  $c = 0$ , the sequence  $z_{n+1} = -4z_n^2 + 4z_n$  is conjugate to the sequence

$$w_{n+1} = w_n^2 + d,$$

where  $d = ac + \frac{1}{2}b - \frac{1}{4}b^2 = 2 - 4 = -2$ , using

$$h(z) = -4z + 2. \text{ In this case, } w_0 = h\left(\frac{1}{2}\right) = 0.$$

**(b)** By Theorem 2.1, with  $a = -2$ ,  $b = 0$ ,  $c = 1$ , the sequence  $z_{n+1} = 1 - 2z_n^2$  is conjugate to the sequence

$$w_{n+1} = w_n^2 + d,$$

where  $d = ac + \frac{1}{2}b - \frac{1}{4}b^2 = -2$ , using  $h(z) = -2z$ . In this case,  $w_0 = h(0) = 0$ .

**2.2 (a)**  $P_c^2(z) = P_c(P_c(z))$

$$= (z^2 + c)^2 + c$$

$$= z^4 + 2cz^2 + c^2 + c;$$

$$P_c^3(z) = (z^4 + 2cz^2 + c^2 + c)^2 + c$$

$$= z^8 + 4cz^6 + (6c^2 + 2c)z^4 + (4c^3 + 4c^2)z^2 + c^4 + 2c^3 + c^2 + c.$$

(b) We have

$$\begin{aligned} P_c^{n+1}(z) &= P_c(P_c^n(z)) \\ &= (P_c^n(z))^2 + c. \end{aligned} \quad (1)$$

Since  $P_c(z) = z^2 + c$ , it follows from Equation (1) that the degree of  $P_c^{n+1}$  is twice the degree of  $P_c^n$ , and hence the degree of  $P_c^n$  is  $2^n$ . The evenness of  $P_c^n$  follows from the fact that  $P_c$  is even, because

$$P_c(-z) = (-z)^2 + c = z^2 + c = P_c(z).$$

**2.3 (a)** The fixed point equation is  $P_c(z) = z^2 + c = z$ , and

$$z^2 + c = z \iff z^2 - z + c = 0.$$

Thus the fixed points are

$$\frac{1 \pm \sqrt{1-4c}}{2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - c}.$$

Let these fixed points be  $\alpha$  and  $\beta$ , so that  $\alpha + \beta = 1$ , and

$$P'_c(\alpha) = 2\alpha \text{ and } P'_c(\beta) = 2\beta.$$

Now  $\alpha + \beta = 1$  and so  $\frac{1}{2}(P'_c(\alpha) + P'_c(\beta)) = 1$ . Unless  $c = \frac{1}{4}$  (so that  $P'_c(\alpha) = P'_c(\beta) = 1$ ), this implies that at least one of  $P'_c(\alpha)$ ,  $P'_c(\beta)$  lies outside the unit circle, and so the corresponding fixed point is repelling.

(In fact, if  $\alpha = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$ , then  $P'_c(\alpha)$  lies outside the unit circle.)

(b) If  $c = \frac{1}{4}$ , then  $P_c = P_{1/4}$  has just one fixed point,  $\frac{1}{2}$ , which is indifferent since  $P'_{1/4}(\frac{1}{2}) = 1$ .

**2.4 (a)**  $r_0 = \frac{1}{2} + \sqrt{\frac{1}{4} + 0} = \frac{1}{2} + \frac{1}{2} = 1$ ;

$$r_i = \frac{1}{2} + \sqrt{\frac{1}{4} + |i|} = \frac{1}{2} + \sqrt{\frac{5}{4}} = \frac{1}{2}(1 + \sqrt{5});$$

$$r_{-2} = \frac{1}{2} + \sqrt{\frac{1}{4} + |-2|} = \frac{1}{2} + \sqrt{\frac{9}{4}} = 2.$$

(b) The sequence  $\{P_0^n(1)\}$  has terms  $1, 1, 1, \dots$ , and so  $P_0^n(1) \rightarrow \infty$  as  $n \rightarrow \infty$ . This shows that  $r_0 = 1$  is the smallest value of  $r_0$  for which Theorem 2.2 holds with  $c = 0$ .

The sequence  $\{P_{-2}^n(2)\}$  has terms  $2, 2, 2, \dots$ , and so  $P_{-2}^n(2) \rightarrow \infty$  as  $n \rightarrow \infty$ . This shows that  $r_{-2} = 2$  is the smallest value of  $r_{-2}$  for which Theorem 2.2 holds with  $c = -2$ .

**2.5 (a)** If  $z_0 \in L$ , then  $z_0$  is real with  $|z_0| \leq 2$ , and so

$$z_1 = z_0^2 - 2$$

is real with  $-2 \leq z_1 \leq 2$ , because  $0 \leq z_0^2 \leq 4$ ; hence  $z_1 \in L$ . On repeating this process, we deduce that  $z_n \in L$ , for  $n = 1, 2, \dots$ .

If  $z_0 \in \mathbb{C} - L$ , then  $z_1 \in \mathbb{C} - L$ . For if  $z_1 \in L$ , then

$$z_0^2 = z_1 + 2 \in [0, 4] \implies z_0 \in L.$$

On repeating this process, we deduce that

$$z_n \in \mathbb{C} - L, \text{ for } n = 1, 2, \dots$$

(b) Let  $w_n = J^{-1}(z_n)$ , for  $n = 0, 1, 2, \dots$ , so that

$$|w_n| > 1, \text{ for } n = 0, 1, 2, \dots$$

Then  $z_n = J(w_n)$ , and so, for  $n = 0, 1, 2, \dots$ , the equation  $z_{n+1} = z_n^2 - 2$  gives

$$\begin{aligned} J(w_{n+1}) &= (J(w_n))^2 - 2 \\ &= \left(w_n + \frac{1}{w_n}\right)^2 - 2 \\ &= w_n^2 + \frac{1}{w_n^2} = J(w_n^2). \end{aligned}$$

Now the function  $J$  is one-one on  $\{w : |w| > 1\}$  and hence

$$w_{n+1} = w_n^2, \text{ for } n = 0, 1, 2, \dots$$

Since  $|w_0| > 1$ , we have  $w_n \rightarrow \infty$  as  $n \rightarrow \infty$ , and hence  $z_n = w_n + 1/w_n \rightarrow \infty$  as  $n \rightarrow \infty$  (by the Reciprocal Rule).

(c) By part (a), we find that no point of  $L$  belongs to  $E_{-2}$ , whereas by part (b) all points of  $\mathbb{C} - L$  belong to  $E_{-2}$ . Hence  $E_{-2} = \mathbb{C} - L$ , and  $K_{-2} = \mathbb{C} - E_{-2} = L$ .

**2.6** To prove that  $E_c$  is completely invariant under  $P_c$ , we note that

$$\begin{aligned} z \in E_c &\iff P_c^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty \\ &\iff P_c^{n+1}(z) \rightarrow \infty \text{ as } n \rightarrow \infty \\ &\iff P_c^n(P_c(z)) \rightarrow \infty \text{ as } n \rightarrow \infty \\ &\iff P_c(z) \in E_c, \end{aligned}$$

as required.

**2.7 (a)** Since

$$P_i(-i) = (-i)^2 + i = -1 + i$$

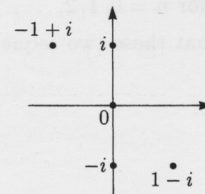
and

$$P_i(-1 + i) = (-1 + i)^2 + i = -i,$$

we have

$$P_i^2(-i) = -i \text{ and } P_i^2(-1 + i) = -1 + i,$$

and so  $-i$  and  $-1 + i$  form a 2-cycle of  $P_i$ . Hence both these points lie in  $K_i$  as do  $i$  and  $1 - i$ , by Theorem 2.3(e). Another point lying in  $K_i$  is 0, because  $P_i(0) = i$  and  $i \in K_i$ .



Clearly none of these points are fixed points of  $K_i$ .

(b) Since  $P_0^3(z) = z^8$ , we have to solve the equation  $z^8 = z$ :

$$\begin{aligned} z^8 = z &\iff z^8 - z = 0 \\ &\iff z(z^7 - 1) = 0. \end{aligned}$$

The solutions are 0 and  $e^{2\pi ki/7}$ ,  $k = 0, 1, \dots, 6$  (Unit A1, Theorem 3.1).

Of these, the points 0 and 1 ( $k = 0$ ) are fixed points of  $P_0$ , whereas

$$\begin{aligned} P_0(e^{2\pi i/7}) &= e^{4\pi i/7}, \\ P_0(e^{4\pi i/7}) &= e^{8\pi i/7}, \\ P_0(e^{8\pi i/7}) &= e^{16\pi i/7} = e^{2\pi i/7}, \end{aligned}$$

and

$$\begin{aligned} P_0(e^{6\pi i/7}) &= e^{12\pi i/7}, \\ P_0(e^{12\pi i/7}) &= e^{24\pi i/7} = e^{10\pi i/7}, \\ P_0(e^{10\pi i/7}) &= e^{20\pi i/7} = e^{6\pi i/7}. \end{aligned}$$

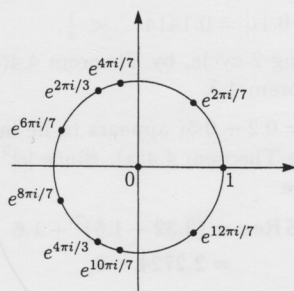
Hence

$$e^{2\pi i/7}, e^{4\pi i/7}, e^{8\pi i/7} \text{ and } e^{6\pi i/7}, e^{12\pi i/7}, e^{10\pi i/7}$$

are both 3-cycles of  $P_0$ .



Using this and the results of Example 2.1, we obtain the following diagram.



(c) Since

$$\begin{aligned} P_{-5/4}\left(\frac{1}{2}(-1 + \sqrt{2})\right) &= \left(\frac{1}{2}(-1 + \sqrt{2})\right)^2 - \frac{5}{4} \\ &= \frac{1}{4}(3 - 2\sqrt{2}) - \frac{5}{4} \\ &= \frac{1}{2}(-1 - \sqrt{2}) \\ &\neq \frac{1}{2}(-1 + \sqrt{2}), \end{aligned}$$

and

$$\begin{aligned} P_{-5/4}^2\left(\frac{1}{2}(-1 + \sqrt{2})\right) &= P_{-5/4}\left(\frac{1}{2}(-1 - \sqrt{2})\right) \\ &= \left(\frac{1}{2}(-1 - \sqrt{2})\right)^2 - \frac{5}{4} \\ &= \frac{1}{4}(3 + 2\sqrt{2}) - \frac{5}{4} \\ &= \frac{1}{2}(-1 + \sqrt{2}), \end{aligned}$$

the point  $\frac{1}{2}(-1 + \sqrt{2})$  is a periodic point, with period 2, of  $P_{-5/4}$ . (It forms a 2-cycle with  $\frac{1}{2}(-1 - \sqrt{2})$ .)

**2.8 (a)** The numbers  $-i$ ,  $-1 + i$  form a 2-cycle of  $P_i$ , with multiplier

$$\begin{aligned} (P_i^2)'(-i) &= P_i'(-i)P_i'(-1 + i) \\ &= (-2i)(-2 + 2i) \\ &= 4 + 4i, \end{aligned}$$

by Theorem 2.4. Hence

$$\left|(P_i^2)'(-i)\right| = 4\sqrt{2} > 1,$$

and so this 2-cycle is repelling.

(b) The numbers  $e^{2\pi i/7}$ ,  $e^{4\pi i/7}$ ,  $e^{8\pi i/7}$  form a 3-cycle of  $P_0$ , with multiplier

$$\begin{aligned} (P_0^3)'(e^{2\pi i/7}) &= P_0'(e^{2\pi i/7})P_0'(e^{4\pi i/7})P_0'(e^{8\pi i/7}) \\ &= (2e^{2\pi i/7})(2e^{4\pi i/7})(2e^{8\pi i/7}) \\ &= 8, \end{aligned}$$

by Theorem 2.4. Hence

$$\left|(P_0^3)'(e^{2\pi i/7})\right| > 1,$$

and so this 3-cycle is repelling.

(c) The numbers  $\frac{1}{2}(-1 + \sqrt{2})$ ,  $\frac{1}{2}(-1 - \sqrt{2})$  form a 2-cycle of  $P_{-5/4}$ , with multiplier

$$\begin{aligned} (P_{-5/4}^2)'(\tfrac{1}{2}(-1 + \sqrt{2})) &= P_{-5/4}'(\tfrac{1}{2}(-1 + \sqrt{2})) \\ &\quad \times P_{-5/4}'(\tfrac{1}{2}(-1 - \sqrt{2})) \\ &= (-1 + \sqrt{2})(-1 - \sqrt{2}) \\ &= -1, \end{aligned}$$

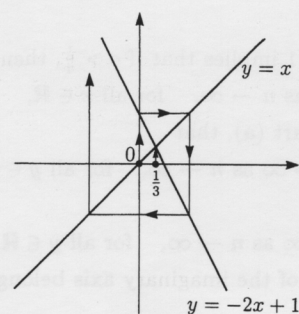
by Theorem 2.4. Hence

$$\left|(P_{-5/4}^2)'(\tfrac{1}{2}(-1 + \sqrt{2}))\right| = 1,$$

and so this 2-cycle is indifferent.

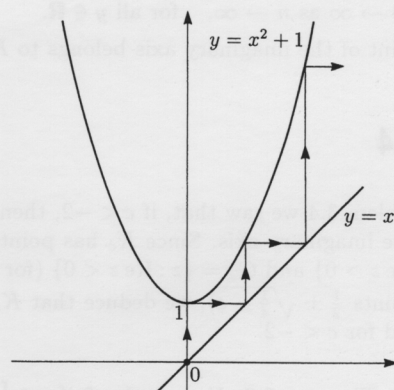
## Section 3

### 3.1 (a)



(b) If  $x_0 = 0$ , then  $\{x_n\}$  tends to infinity and if  $x_0 = \frac{1}{3}$ , then the sequence is constant. In Problem 1.9(b)(iii) we found that  $z_{n+1} = az_n + b$ ,  $n = 0, 1, 2, \dots$ , tends to infinity if  $|a| > 1$  unless  $z_0 = b/(1 - a)$ . Here we have  $a = -2$ ,  $b = 1$  and  $b/(1 - a) = \frac{1}{3}$ , and so our answers agree with this result.

**3.2 (a)** With  $x_0 = 0$ , graphical iteration gives the following diagram.



(b) Since

$$\begin{aligned} x^2 + 1 > x &\iff x^2 - x + 1 > 0 \\ &\iff \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0, \end{aligned}$$

we deduce that

$$x^2 + 1 > x, \quad \text{for all } x \in \mathbb{R}.$$

Hence the graph  $y = x^2 + 1$  lies strictly above  $y = x$ , that is, the function  $f(x) = x^2 + 1$  has no real fixed points. Also  $x_{n+1} = x_n^2 + 1 > x_n$  and so the sequence  $\{x_n\}$  is increasing and must tend to infinity (because there are no fixed points to prevent this).

(c) Since  $x_n = P_1^n(x_0)$ , for  $n = 1, 2, \dots$ , we deduce that

$$P_1^n(x_0) \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \text{for all } x_0 \in \mathbb{R},$$

and so no point of  $\mathbb{R}$  belongs to  $K_1$ .

**3.3 (a)** If  $c$  is real and  $y$  is real, then

$$P_c(iy) = (iy)^2 + c = -y^2 + c,$$

which is real.

**(b)** Theorem 3.1 implies that if  $c > \frac{1}{4}$ , then

$$P_c^n(x) \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ for all } x \in \mathbb{R},$$

and hence, by part (a), that

$$P_c^n(P_c(iy)) \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ for all } y \in \mathbb{R}.$$

Thus

$$P_c^{n+1}(iy) \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ for all } y \in \mathbb{R},$$

and so no point of the imaginary axis belongs to  $K_c$ .

**3.4** If  $c < -2$  and  $y \in \mathbb{R}$ , then

$$P_c(iy) = -y^2 + c$$

$$\leq c$$

$$< -\frac{1}{2} - \sqrt{\frac{1}{4} - c}$$

and so  $P_c(iy)$  lies outside the interval  $I_c$  (see Figure 3.7). Hence

$$P_c^n(P_c(iy)) \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ for all } y \in \mathbb{R},$$

by (3.3), and so

$$P_c^{n+1}(iy) \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ for all } y \in \mathbb{R}.$$

Thus no point of the imaginary axis belongs to  $K_c$ .

## Section 4

**4.1** In Problem 3.4 we saw that, if  $c < -2$ , then  $K_c$  does not meet the imaginary axis. Since  $K_c$  has points in both  $G_1 = \{z : \operatorname{Re} z > 0\}$  and  $G_2 = \{z : \operatorname{Re} z < 0\}$  (for example, the fixed points  $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$ ), we deduce that  $K_c$  is disconnected for  $c < -2$ .

**4.2 (a)** By Theorem 3.2,  $K_c$  contains 0 if  $c \in [-2, \frac{1}{4}]$  and so, by Theorem 4.1,  $K_c$  is connected and hence  $c \in M$ .

**(b)** Since  $K_c$  is disconnected for  $c > \frac{1}{4}$  and  $c < -2$ , by Problem 4.1 and the discussion preceding it, we have  $c \notin M$  for  $c > \frac{1}{4}$  and  $c < -2$ . Hence, by part (a),  $M \cap \mathbb{R} = [-2, \frac{1}{4}]$ .

**4.3 (a)** If  $c = -2$ , then the terms of  $\{P_c^n(0)\}$  are  $-2, 2, 2, \dots$ .

Since all these terms lie in  $\{z : |z| \leq 2\}$ , we deduce by Corollary 1 that  $-2 \in M$ .

**(b)** If  $c = 1 + i$ , then the terms of  $\{P_c^n(0)\}$  are

$$1 + i, (1 + i)^2 + 1 + i = 1 + 3i, \dots$$

Since  $|1 + 3i| = \sqrt{10} > 2$ , we deduce by Corollary 1 that  $1 + i \notin M$ .

**(c)** If  $c = i$ , then the terms of  $\{P_c^n(0)\}$  are

$$i, -1 + i, -i, -1 + i, \dots$$

Since all these terms lie in  $\{z : |z| \leq 2\}$ , we deduce by Corollary 1 that  $i \in M$ .

**(d)** If  $c = \sqrt{2}i$ , then the terms of  $\{P_c^n(0)\}$  are

$$\sqrt{2}i, -2 + \sqrt{2}i, \dots$$

Since  $|-2 + \sqrt{2}i| = \sqrt{6} > 2$ , we deduce by Corollary 1 that  $\sqrt{2}i \notin M$ .

**4.4 (a)** The point  $c = -0.9 + 0.1i$  appears to lie in the disc  $|z + 1| < \frac{1}{4}$ , so we use Theorem 4.4(b). Since

$$|c + 1| = |0.1 + 0.1i| = 0.1414 \dots < \frac{1}{4},$$

$P_c$  has an attracting 2-cycle, by Theorem 4.4(b). Thus  $c$  lies in  $M$ , by Theorem 4.3.

**(b)** The point  $c = 0.2 + 0.5i$  appears to lie inside the cardioid, so we use Theorem 4.4(a). Since  $|c|^2 = 0.29$  and  $\operatorname{Re} c = 0.2$ , we have

$$\begin{aligned} (8|c|^2 - \frac{3}{2})^2 + 8\operatorname{Re} c &= (2.32 - 1.5)^2 + 1.6 \\ &= 2.2724 < 3, \end{aligned}$$

and so  $P_c$  has an attracting fixed point, by Theorem 4.4(a). Thus  $c$  lies in  $M$ , by Theorem 4.3.

**4.5 (a)** Since  $P_c^2(z) = (z^2 + c)^2 + c$ , we have

$$P_c^2(z) - z = z^4 + 2cz^2 - z + c^2 + c.$$

Also

$$\begin{aligned} (P_c(z) - z)(z^2 + z + c + 1) &= (z^2 - z + c)(z^2 + z + c + 1) \\ &= z^4 + 2cz^2 - z + c^2 + c, \end{aligned}$$

as required.

**(b)** The 2-cycles of  $P_c$  are the solutions of  $P_c^2(z) - z = 0$  which are not solutions of  $P_c(z) - z = 0$ . Hence, by part (a), they are the solutions of

$$z^2 + z + c + 1 = 0;$$

which gives the 2-cycle  $\alpha_1, \alpha_2$  where

$$\alpha_1 = -\frac{1}{2} + \sqrt{-\frac{3}{4} - c}, \quad \alpha_2 = -\frac{1}{2} - \sqrt{-\frac{3}{4} - c}.$$

Note that  $c = -\frac{3}{4}$  must be excluded, because in that case  $\alpha_1 = \alpha_2 = -\frac{1}{2}$ , which is a fixed point of  $P_{-3/4}$ . By Theorem 2.4, the multiplier of this 2-cycle is

$$\begin{aligned} (P_c^2)'(\alpha_1) &= P_c'(\alpha_1)P_c'(\alpha_2) \\ &= (2\alpha_1)(2\alpha_2) \\ &= 4\alpha_1\alpha_2. \end{aligned}$$

**(c)** Since  $\alpha_1\alpha_2 = c + 1$ , we deduce that the above 2-cycle is attracting if and only if

$$|(P_c^2)'(\alpha_1)| = |4\alpha_1\alpha_2| = 4|c + 1| < 1;$$

that is, if and only if  $|c + 1| < \frac{1}{4}$ , as required.

**4.6** The function  $P_c$  has a super-attracting 3-cycle if and only if

$$P_c^3(0) = (c^2 + c)^2 + c = 0,$$

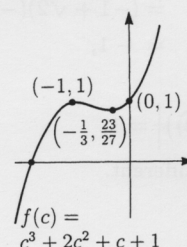
but  $P_c(0) = c \neq 0$ ,  $P_c^2(0) = c^2 + c \neq 0$ . Thus we seek the solutions of

$$c^4 + 2c^3 + c^2 + c = 0$$

which are not  $c = 0$  nor  $c = -1$ . Since  $c = 0$  is a solution and  $c = -1$  is not, we need to solve

$$c^3 + 2c^2 + c + 1 = 0.$$

Using calculus, we see that the real function  $f(c) = c^3 + 2c^2 + c + 1$  has the following graph, so that there is only one real zero.





Since  $f(-1.8) = -0.152$  and  $f(-1.7) = 0.167$ , it follows that this value of  $c$  lies in  $[-1.8, -1.7]$ .

The remaining two solutions form a pair of complex conjugates. (In fact the three solutions, correct to three decimal places, are  $-1.755$  and  $-0.123 \pm 0.745i$ . (See Figure 2.18(b)!) These solutions may be obtained by using Formula (0.6) in the Introduction to *Unit A1*!)

**4.7 (a)** Since any solution of  $P_c(z) - z = 0$  is also a solution of  $P_c^3(z) - z = 0$ , we expect  $P_c(z) - z$  to be a factor of  $P_c^3(z) - z = 0$ . By inspection, we find that

$$\begin{aligned} P_c^3(z) - z &= (P_c(z) - z)Q_c(z) \\ &= (z^2 - z + c)Q_c(z), \end{aligned}$$

where

$$\begin{aligned} Q_c(z) &= z^6 + z^5 + (3c+1)z^4 + (2c+1)z^3 \\ &\quad + (3c^2+3c+1)z^2 + (c^2+2c+1)z \\ &\quad + c^3 + 2c^2 + c + 1. \end{aligned}$$

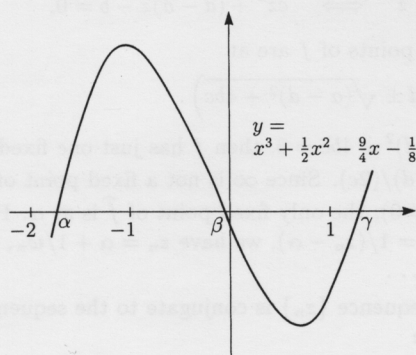
(Begin by finding the coefficient of  $z^6$  and the constant term, and then work inwards from both ends, obtaining the coefficient of  $z^3$  twice, as a check.)

**(b)** On substituting  $c = -\frac{7}{4}$  we obtain, after some arithmetic,

$$\begin{aligned} Q_{-7/4}(z) &= z^6 + z^5 - \frac{17}{4}z^4 - \frac{5}{2}z^3 + \frac{79}{16}z^2 + \frac{9}{16}z + \frac{1}{64} \\ &= (z^3 + \frac{1}{2}z^2 - \frac{9}{4}z - \frac{1}{8})^2, \end{aligned}$$

by inspection again.

**(c)** The graph  $y = x^3 + \frac{1}{2}x^2 - \frac{9}{4}x - \frac{1}{8}$  is as follows.



Hence

$$z^3 + \frac{1}{2}z^2 - \frac{9}{4}z - \frac{1}{8} = (z - \alpha)(z - \beta)(z - \gamma), \quad (1)$$

where  $\alpha, \beta, \gamma$  are the real zeros in the figure. Since these are the only solutions of  $P_{-7/4}^3(z) - z = 0$ , which are not fixed points of  $P_{-7/4}$ , they must form a 3-cycle of  $P_{-7/4}$ . By Theorem 2.4, the multiplier of this 3-cycle is

$$\begin{aligned} (P_c^3)'(\alpha) &= P_c'(\alpha)P_c'(\beta)P_c'(\gamma) \\ &= (2\alpha)(2\beta)(2\gamma) \\ &= 8\alpha\beta\gamma \\ &= 1, \end{aligned}$$

since  $\alpha\beta\gamma = \frac{1}{8}$ , by Equation (1).

Hence, by Theorem 4.6(a), a saddle-node bifurcation occurs at  $c = -7/4$ , so we expect to see a small cardioid-shaped periodic region with cusp at this point. This is included in Figure 4.8, and its centre is at the point  $c = -1.755$ , found in Problem 4.6.

**4.8** Since

$$P_c(\zeta) = \zeta^2 + c = \zeta \quad (c = \zeta - \zeta^2),$$

we find that  $\zeta$  is a fixed point of  $P_c$ . Also, the multiplier is

$$P_c'(\zeta) = 2\zeta,$$

which is a root of unity ( $\neq 1$ ). Hence, by Theorem 4.6(b), a period-multiplying bifurcation occurs at  $c$ . (In fact, since the cardioid is the image of the circle  $|z| = \frac{1}{2}$  under the function  $f(z) = z - z^2$ , the point  $c$  lies on the cardioid.)

If  $\zeta = -\frac{1}{2}$ , then  $2\zeta = -1$ , which is a primitive square root of unity and so a period-doubling bifurcation occurs at

$$c = -\frac{1}{2} - \left(-\frac{1}{2}\right)^2 = -\frac{3}{4},$$

and this is visible in Figure 4.8.

If  $\zeta = \frac{1}{2}e^{2\pi i/3}$ , then  $2\zeta = e^{2\pi i/3}$ , which is a primitive cube root of unity and so a 'period-trebling' bifurcation occurs at

$$c = \frac{1}{2}e^{2\pi i/3} - \left(\frac{1}{2}e^{2\pi i/3}\right)^2 = -\frac{1}{8} + \frac{3}{8}\sqrt{3}i,$$

and this is visible in Figure 4.8.

If  $\zeta = \frac{1}{2}i$ , then  $2\zeta = i$ , which is a primitive fourth root of unity and so a 'period-quadrupling' bifurcation occurs at

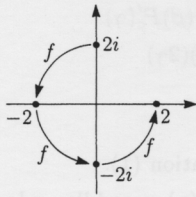
$$c = \frac{1}{2}i - \left(\frac{1}{2}i\right)^2 = \frac{1}{4} + \frac{1}{2}i,$$

and this is visible in Figure 4.8.

# SOLUTIONS TO THE EXERCISES

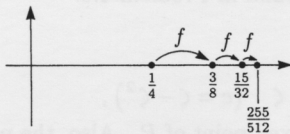
## Section 1

1.1 (a)  $z_0 = 2i, z_1 = -2, z_2 = -2i, z_3 = 2$ .



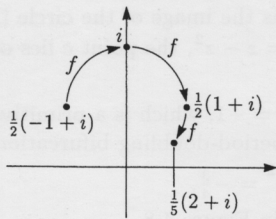
Here  $f(z) = iz$ .

(b)  $z_0 = \frac{1}{4}, z_1 = \frac{3}{8}, z_2 = \frac{15}{32}, z_3 = \frac{255}{512}$ .



Here  $f(z) = 2z(1 - z)$ .

(c)  $z_0 = \frac{1}{2}(-1 + i), z_1 = i, z_2 = \frac{1}{2}(1 + i), z_3 = \frac{1}{5}(2 + i)$ .



Here  $f(z) = \frac{z}{z+1}$ .

1.2 (a) We have to solve  $f(z) = z$ :

$$\begin{aligned} z - z^2 &= z &\iff z^2 = 0 \\ &&\iff z = 0. \end{aligned}$$

Since  $f'(z) = 1 - 2z$ , we have

$$f'(0) = 1,$$

so the fixed point at 0 is indifferent.

(b) We have to solve  $f(z) = z$ :

$$2z(1 - z) = z \iff z(1 - 2z) = 0,$$

so the fixed points of  $f$  are at 0 and  $\frac{1}{2}$ . Since

$f'(z) = 2 - 4z$ , we have

$$f'(0) = 2 \quad \text{and} \quad f'\left(\frac{1}{2}\right) = 0.$$

Thus the fixed point at 0 is repelling, whereas that at  $\frac{1}{2}$  is super-attracting.

(c) We have to solve  $f(z) = z$ :

$$z^2 - \frac{1}{2} = z \iff z^2 - z - \frac{1}{2} = 0,$$

so the fixed points of  $f$  are at  $\frac{1}{2}(1 \pm \sqrt{3})$ . Since

$f'(z) = 2z$ , we have

$$\left|f'\left(\frac{1}{2}(1 + \sqrt{3})\right)\right| = 1 + \sqrt{3} = 2.732 \dots$$

and

$$\left|f'\left(\frac{1}{2}(1 - \sqrt{3})\right)\right| = 0.732 \dots$$

Thus the fixed point of  $f$  at  $\frac{1}{2}(1 + \sqrt{3})$  is repelling, whereas that at  $\frac{1}{2}(1 - \sqrt{3})$  is attracting.

(d) We have to solve  $f(z) = z$ :

$$\frac{z}{z+1} = z \iff z = 0,$$

so the only fixed point of  $f$  is at 0. Since

$$f'(z) = 1/(z+1)^2, \text{ we have}$$

$$f'(0) = 1.$$

Thus the fixed point of  $f$  at 0 is indifferent.

1.3 Putting  $w_n = h(z_n) = 1 - 2z_n$  for  $n = 0, 1, 2, \dots$ , we have

$$z_n = \frac{1}{2}(1 - w_n), \quad \text{for } n = 0, 1, 2, \dots$$

Hence the given sequence is conjugate to the sequence  $\{w_n\}$ , where

$$\begin{aligned} \frac{1}{2}(1 - w_{n+1}) &= 2\left(\frac{1}{2}(1 - w_n)\right)\left(1 - \frac{1}{2}(1 - w_n)\right) \\ &= \frac{1}{2}(1 - w_n)(1 + w_n) \\ &= \frac{1}{2}(1 - w_n^2), \end{aligned}$$

so that  $w_{n+1} = w_n^2$ , for  $n = 0, 1, 2, \dots$ , as required.

Since  $w_n = w_0^{2^n}$ , for  $n = 0, 1, 2, \dots$ , by Example 1.2(b), we deduce that

$$\begin{aligned} z_n &= \frac{1}{2}(1 - w_n) \\ &= \frac{1}{2}(1 - w_0^{2^n}) \\ &= \frac{1}{2}\left(1 - (1 - 2z_0)^{2^n}\right), \quad \text{for } n = 0, 1, 2, \dots \end{aligned}$$

1.4 To find the fixed points of  $f$  we solve  $f(z) = z$ :

$$\frac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0,$$

so the fixed points of  $f$  are at

$$\frac{1}{2c}\left(a - d \pm \sqrt{(a-d)^2 + 4bc}\right).$$

(a) If  $(a-d)^2 + 4bc = 0$ , then  $f$  has just one fixed point at  $\alpha = (a-d)/(2c)$ . Since  $\infty$  is not a fixed point of  $\hat{f}$  (because  $c \neq 0$ ), the only fixed point of  $\hat{f}$  is at  $\alpha$ . Putting  $w_n = h(z_n) = 1/(z_n - \alpha)$ , we have  $z_n = \alpha + 1/w_n$ , for  $n = 0, 1, 2, \dots$ .

Hence the sequence  $\{z_n\}$  is conjugate to the sequence  $\{w_n\}$ , where

$$\begin{aligned} \alpha + 1/w_{n+1} &= \frac{a(\alpha + 1/w_n) + b}{c(\alpha + 1/w_n) + d} \\ &= \frac{a\alpha + b + a/w_n}{c\alpha + d + c/w_n} \\ &= \frac{(a\alpha + b)w_n + a}{(c\alpha + d)w_n + c}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{w_{n+1}} &= \frac{(a\alpha + b)w_n + a - \alpha(c\alpha + d)w_n - \alpha c}{(c\alpha + d)w_n + c} \\ &= \frac{a - \alpha c}{(c\alpha + d)w_n + c} \quad \left(\text{since } \frac{a\alpha + b}{c\alpha + d} = \alpha\right), \end{aligned}$$

and so

$$\begin{aligned} w_{n+1} &= \left(\frac{c\alpha + d}{a - \alpha c}\right)w_n + \frac{c}{a - \alpha c} \\ &= w_n + 2c/(a + d), \quad \text{for } n = 0, 1, 2, \dots, \end{aligned} \quad (1)$$

because  $c\alpha + d = \frac{1}{2}(a + d) = a - \alpha c$ . Note that  $c \neq 0$  and also  $a + d \neq 0$ , because

$$\begin{aligned} (a + d)^2 &= (a - d)^2 + 4bc + 4(ad - bc) \\ &= 4(ad - bc) \neq 0. \end{aligned}$$



It follows from (1) that  $w_n \rightarrow \infty$  as  $n \rightarrow \infty$ , for all  $w_0 \in \widehat{\mathbb{C}}$  and hence

$$z_n = \alpha + \frac{1}{w_n} \rightarrow \alpha \text{ as } n \rightarrow \infty, \text{ for all } z_0 \in \widehat{\mathbb{C}}.$$

*Remark* The sequence  $w_n$  tends to infinity along a straight line, and so  $z_n$  tends to  $\alpha$  along a circle; see the solution to Exercise 1.1(c).

(b) If  $(a-d)^2 + 4bc \neq 0$ , then  $f$  has the two fixed points

$$\alpha = \frac{1}{2c} \left( a - d + \sqrt{(a-d)^2 + 4bc} \right)$$

and

$$\beta = \frac{1}{2c} \left( a - d - \sqrt{(a-d)^2 + 4bc} \right).$$

Putting  $w_n = h(z_n) = (z_n - \alpha)/(z_n - \beta)$ , we have  $z_n = (-\beta w_n + \alpha)/(-w_n + 1)$ , for  $n = 0, 1, 2, \dots$ . Hence the sequence  $\{z_n\}$  is conjugate to the sequence

$$\begin{aligned} w_{n+1} &= h(z_{n+1}) \\ &= h(f(z_n)) \\ &= (h \circ f \circ h^{-1})(w_n). \end{aligned}$$

Composing the Möbius transformations  $h$ ,  $f$  and  $h^{-1}$  (for example, by multiplying the corresponding  $2 \times 2$  matrices), we obtain

$$w_{n+1} = \frac{(-a\beta - b + c\alpha\beta + d\alpha)w_n + (a\alpha + b - c\alpha^2 - d\alpha)}{(-a\beta - b + c\beta^2 + d\beta)w_n + (a\alpha + b - c\alpha\beta - d\beta)}.$$

Since  $a\alpha + b = \alpha(c\alpha + d)$  and  $a\beta + b = \beta(c\beta + d)$ , it follows that

$$\begin{aligned} w_{n+1} &= \frac{(\alpha - \beta)(c\beta + d)w_n + 0}{0 + (\alpha - \beta)(c\alpha + d)} \\ &= \left( \frac{c\beta + d}{c\alpha + d} \right) w_n, \text{ for } n = 0, 1, 2, \dots \end{aligned}$$

Now

$$f'(\alpha) = \frac{ad - bc}{(c\alpha + d)^2},$$

and so to obtain  $w_{n+1} = f'(\alpha)w_n$ , for  $n = 0, 1, 2, \dots$ , we need to check that  $ad - bc = (c\alpha + d)(c\beta + d)$ . Since

$$c\alpha + d = \frac{1}{2} \left( a + d + \sqrt{(a-d)^2 + 4bc} \right)$$

and

$$c\beta + d = \frac{1}{2} \left( a + d - \sqrt{(a-d)^2 + 4bc} \right),$$

we obtain

$$\begin{aligned} (c\alpha + d)(c\beta + d) &= \frac{1}{4} \left( (a+d)^2 - (a-d)^2 - 4bc \right) \\ &= ad - bc, \end{aligned}$$

as required.

If  $|f'(\alpha)| < 1$ , then  $w_n \rightarrow 0$  as  $n \rightarrow \infty$ , for all  $w_0 \in \mathbb{C}$ , and hence

$$z_n = \frac{-\beta w_n + \alpha}{-w_n + 1} \rightarrow \alpha \text{ as } n \rightarrow \infty,$$

for all  $z_0 \in \widehat{\mathbb{C}} - \{\beta\}$ .

If  $|f'(\alpha)| > 1$ , then  $w_n \rightarrow \infty$  as  $n \rightarrow \infty$ , for all  $w_0 \in \mathbb{C} - \{0\}$  and hence

$$z_n = \frac{-\beta w_n + \alpha}{-w_n + 1} \rightarrow \beta \text{ as } n \rightarrow \infty,$$

for all  $z_0 \in \widehat{\mathbb{C}} - \{\alpha\}$ .

If  $|f'(\alpha)| = 1$ , then  $|w_n| = |w_0|$ , for  $n = 1, 2, \dots$ , and so the sequence  $\{w_n\}$  remains on the circle with centre 0 and radius  $|w_0|$ . Hence the sequence  $\{z_n\}$  remains on the image of this circle under  $h^{-1}$ , which is a generalized circle with  $\alpha$  and  $\beta$  as inverse points.

## Section 2

**2.1** Using Theorem 2.1, we find that

$$z_{n+1} = 3z_n(1 - z_n) = -3z_n^2 + 3z_n, \quad n = 0, 1, 2, \dots,$$

is conjugate to

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \dots,$$

where  $w_n = -3z_n + 3/2$ , for  $n = 0, 1, 2, \dots$ , and

$$d = -3 \times 0 + 3/2 - 9/4 = -3/4.$$

**2.2 (a)** If  $|c| \leq \frac{1}{4}$  and  $|z| \leq \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$ , then

$$\begin{aligned} |P_c(z)| &= |z^2 + c| \\ &\leq |z|^2 + |c| \quad (\text{Triangle Inequality}) \\ &\leq \left( \frac{1}{2} + \sqrt{\frac{1}{4} - |c|} \right)^2 + |c| \\ &= \frac{1}{4} + \sqrt{\frac{1}{4} - |c|} + \frac{1}{4} - |c| + |c| \\ &= \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}, \end{aligned}$$

as required.

(b) If  $|c| \leq \frac{1}{4}$  and  $|z| \leq \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}$ , then, by part (a) applied repeatedly,

$$|P_c^n(z)| \leq \frac{1}{2} + \sqrt{\frac{1}{4} - |c|}, \quad \text{for } n = 0, 1, 2, \dots,$$

so that  $P_c^n(z) \rightarrow \infty$  as  $n \rightarrow \infty$ , and hence  $z \notin E_c$ .

Therefore, if  $|c| \leq \frac{1}{4}$ , then we have

$$\left\{ z : |z| \leq \frac{1}{2} + \sqrt{\frac{1}{4} - |c|} \right\} \subseteq K_c,$$

as required.

(c) If  $|c| \leq \frac{1}{4}$ , then, by part (b) and Theorem 2.3(a),

$$\left\{ z : |z| \leq \frac{1}{2} + \sqrt{\frac{1}{4} - |c|} \right\} \subseteq K_c \subseteq \left\{ z : |z| \leq \frac{1}{2} + \sqrt{\frac{1}{4} + |c|} \right\}.$$

Now, if  $c$  is close to 0, then  $\frac{1}{2} + \sqrt{\frac{1}{4} \pm |c|}$  are both close to 1, and so  $K_c$  is approximately equal to the closed unit disc.

**2.3 (a)** Since

$$f(i) = -i \quad \text{and} \quad f(-i) = i,$$

the point  $\alpha = i$  is periodic with period 2, and belongs to the 2-cycle  $i, -i$ . Since  $f'(z) = -1$ , the multiplier of this 2-cycle is, by Theorem 2.4,

$$f'(i)f'(-i) = (-1) \times (-1) = 1.$$

Thus this 2-cycle is indifferent, and so  $i$  is an indifferent periodic point of  $f$  with period 2.

(b) Since

$$f\left(\frac{1}{2}(-1 + \sqrt{5})\right) = \frac{1}{2}(-1 - \sqrt{5})$$

and

$$f\left(\frac{1}{2}(-1 - \sqrt{5})\right) = \frac{1}{2}(-1 + \sqrt{5}),$$

the point  $\alpha = \frac{1}{2}(-1 + \sqrt{5})$  is periodic with period 2, and belongs to the 2-cycle  $\frac{1}{2}(-1 + \sqrt{5}), \frac{1}{2}(-1 - \sqrt{5})$ . Since  $f'(z) = 2z$ , the multiplier of this 2-cycle is, by Theorem 2.4,

$$\begin{aligned} f'\left(\frac{1}{2}(-1 + \sqrt{5})\right) f'\left(\frac{1}{2}(-1 - \sqrt{5})\right) &= (-1 + \sqrt{5})(-1 - \sqrt{5}) \\ &= -4. \end{aligned}$$

Thus this 2-cycle is repelling, and so  $\frac{1}{2}(-1 + \sqrt{5})$  is a repelling periodic point of  $f$  with period 2.

(c) Since

$$f(0) = i \quad \text{and} \quad f(i) = 0,$$

the point  $\alpha = 0$  is periodic with period 2, and belongs to the 2-cycle  $0, i$ . Since  $f'(z) = 3z^2$ , the multiplier of this 2-cycle is, by Theorem 2.4,

$$f'(0)f'(i) = 0.$$

Thus this 2-cycle is super-attracting, and so 0 is a super-attracting periodic point of  $f$  with period 2.

(d) Since

$$f(e^{\pi i/13}) = e^{3\pi i/13}, \quad f(e^{3\pi i/13}) = e^{9\pi i/13},$$

and

$$f(e^{9\pi i/13}) = e^{27\pi i/13} = e^{\pi i/13},$$

the point  $\alpha = e^{\pi i/13}$  is periodic with period 3, and belongs to the 3-cycle  $e^{\pi i/13}, e^{3\pi i/13}, e^{9\pi i/13}$ . Since  $f'(z) = 3z^2$ , the multiplier of the 3-cycle is, by Theorem 2.4,

$$\begin{aligned} f'(e^{\pi i/13}) f'(e^{3\pi i/13}) f'(e^{9\pi i/13}) \\ &= (3e^{2\pi i/13})(3e^{6\pi i/13})(3e^{18\pi i/13}) \\ &= 27e^{26\pi i/13} \\ &= 27. \end{aligned}$$

Thus the 3-cycle is repelling, and so  $e^{\pi i/13}$  is a repelling periodic point of  $f$  with period 3.

**2.4 (a)** Since  $P_0^n(z) = z^{2^n}$ ,  $\alpha$  is a periodic point of  $P_0$  if and only if it satisfies the equation  $z^{2^p} = z$ , for some positive integer  $p$ . Now

$$z^{2^p} = z \iff z(z^{2^p-1} - 1) = 0.$$

Thus either  $\alpha = 0$ , which is a super-attracting fixed point of  $P_0$ , or  $\alpha$  is a  $(2^p - 1)$ th root of unity. Since  $P_0'(z) = 2z$ , the multiplier of  $\alpha$  has absolute value  $2^q$  for some factor  $q$  of  $p$  and  $\alpha$  is therefore repelling. Thus the repelling periodic points of  $P_0$  are the  $(2^p - 1)$ th roots of unity, for  $p = 1, 2, \dots$

Now, if  $\Gamma$  is an arc of the unit circle which subtends an angle  $\theta > 0$  at 0, then we can choose  $p$  so large that

$$\frac{2\pi}{2^p - 1} < \theta,$$

and this guarantees that at least one of the  $(2^p - 1)$ th roots of unity (which are evenly spaced around the unit circle) lies in  $\Gamma$ .

(b) Since  $P_0^n(z) = z^{2^n}$ ,  $\alpha$  is a backward iterate of 1 under  $P_0$  if and only if it satisfies the equation

$$z^{2^n} = 1,$$

for some positive integer  $n$ , that is, if and only if  $\alpha$  is a  $2^n$ th root of unity for some positive integer  $n$ .

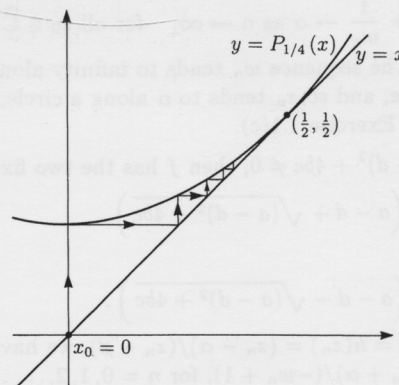
Now, if  $\Gamma$  is an arc of the unit circle which subtends an angle  $\theta > 0$  at 0, then we can choose  $n$  so large that

$$\frac{2\pi}{2^n} < \theta,$$

and this guarantees that at least one of the  $2^n$ th roots of unity lies in  $\Gamma$ .

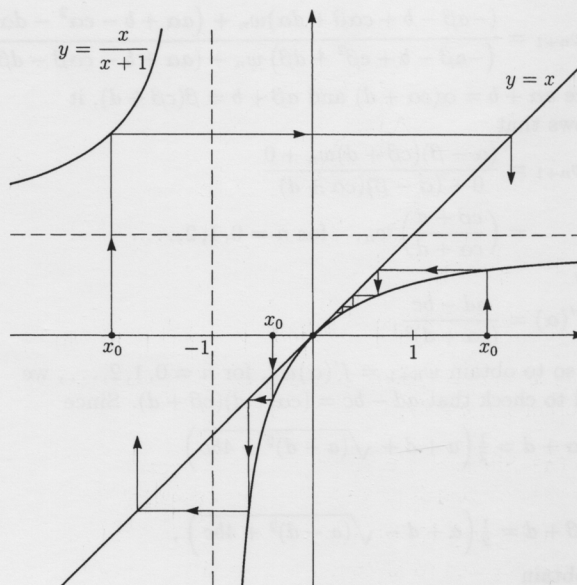
## Section 3

### 3.1



The iteration sequence  $\{x_n\}$  converges to the indifferent fixed point  $\frac{1}{2}$  of  $P_{1/4}$ .

**3.2 (a)** First we plot  $y = x$  and  $y = \frac{x}{x+1}$  on the same diagram, and apply graphical iteration with various initial points  $x_0$ .



Since  $y = x$  lies above  $y = x/(x+1)$ , for  $x > 0$ , graphical iteration shows that if  $x_0 > 0$ , then  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ .

If  $x_0 < -1$ , then  $x_1 = \frac{x_0}{x_0+1} > 0$  and so again  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ . If  $-1 < x_0 < 0$  and  $x_0 \notin A$ , then graphical iteration shows that  $x_{n_0} < -1$  for some positive integer  $n_0$  and hence  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Since  $x_n$  evidently tends to 0 if  $x_0 = 0$ , we deduce that  $x_n \rightarrow 0$  for all initial values  $x_0$  in  $\mathbb{R} - A$ .

(b) The function

$$f(z) = \frac{z}{z+1}$$

is a Möbius transformation, with  $a = 1, b = 0, c = 1, d = 1$ , which satisfies  $(a-d)^2 + 4bc = 0$ . Hence, by Exercise 1.4(a),  $\alpha = 0$  is the only fixed point of  $f$  and the iteration sequence

$$z_{n+1} = \hat{f}(z_n), \quad n = 0, 1, 2, \dots,$$

converges to  $\alpha = 0$  for all initial points  $z_0$  in  $\hat{\mathbb{C}}$ . The result in part (a) is therefore a special case of Exercise 1.4(a).



## Section 4

**4.1 (a)** Since  $|P_c(0)| = |c| = \sqrt{5} > 2$  if  $c = -1 + 2i$ , we deduce, by Corollary 1 to Theorem 4.1, that  $c \notin M$ .

**(b)** Since  $P_{-1}(0) = -1$  and  $P_{-1}(-1) = 0$ , we deduce that the terms of the sequence  $\{P_{-1}^n(0)\}$  are  $0, -1, 0, -1, \dots$ , and so

$$|P_{-1}^n(0)| \leq 2, \text{ for } n = 1, 2, \dots$$

Hence  $-1 \in M$ .

**(c)** For  $c = -1 + i$ ,

$$P_c(0) = -1 + i,$$

$$P_c^2(0) = (-1 + i)^2 + (-1 + i) = -1 - i,$$

$$P_c^3(0) = (-1 - i)^2 + (-1 + i) = -1 + 3i.$$

Since

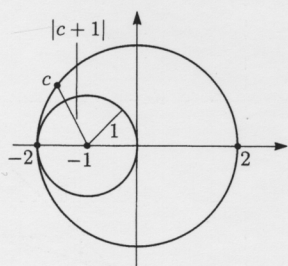
$$|P_c^3(0)| = \sqrt{10} > 2,$$

we deduce, by Corollary 1 to Theorem 4.1, that  $c \notin M$ .

**4.2** Suppose that  $|c| = 2$  but  $c \neq -2$ . Then

$$|c^2 + c| = |c(c+1)| = |c||c+1| = 2|c+1|.$$

Now  $|c+1|$  is the distance from  $c$  to  $-1$ , and so  $|c+1| > 1$ .



Hence  $|c^2 + c| = |P_c^2(0)| > 2$  and so, by Corollary 1 to Theorem 4.1,  $c \notin M$ .

**4.3 (a)** Since  $|c| > r_c$ , if  $|c| > 2$ , we deduce by Theorem 2.2 that if  $|c| > 2$ , then the sequence

$$|P_c^n(c)| = |P_c^{n+1}(0)|, \quad n = 0, 1, 2, \dots,$$

is increasing and so all its terms exceed 2. Hence

$$M_n \subseteq \{c : |c| \leq 2\} = M_1, \quad \text{for } n = 2, 3, \dots$$

**(b)** Since  $r_c \leq 2$ , if  $|c| \leq 2$ , we deduce by Theorem 2.2 that if  $|c| \leq 2$  and  $|P_c^n(0)| > 2$ , then

$$|P_c^{n+1}(0)| = |P_c(P_c^n(0))| \geq |P_c^n(0)| > 2.$$

Thus, for  $|c| \leq 2$ , we deduce that

$$|P_c^{n+1}(0)| \leq 2 \implies |P_c^n(0)| \leq 2, \text{ for } n = 1, 2, \dots,$$

that is,  $M_{n+1} \subseteq M_n$ , for  $n = 1, 2, \dots$ , as required.

**4.4 (a)** If  $c = -1.1 - 0.1i$ , then

$$|c+1| = |-0.1 - 0.1i| = 0.1414 \dots < \frac{1}{4}.$$

Hence, by Theorem 4.4(b),  $P_c$  has an attracting 2-cycle and so, by Theorem 4.3,  $c \in M$ .

**(b)** If  $c = 0.6i$ , then  $|c|^2 = 0.36$  and  $\operatorname{Re} c = 0$ , so that

$$\begin{aligned} \left(8|c|^2 - \frac{3}{2}\right)^2 + 8\operatorname{Re} c &= (8 \times 0.36 - 1.5)^2 \\ &= 1.9044 < 3. \end{aligned}$$

Hence, by Theorem 4.4(a),  $P_c$  has an attracting fixed point and so, by Theorem 4.3,  $c \in M$ .

**4.5 (a)** If some point  $\alpha$ , say, of  $\partial\mathcal{R}$  lies outside  $M$ , then because  $M$  is closed there is an open disc  $D$  with centre  $\alpha$  which lies entirely outside  $M$ . Since  $\mathcal{R} \subseteq M$ , the open disc  $D$  does not meet  $\mathcal{R}$ , and this contradicts the fact that  $\alpha$  is a boundary point of  $\mathcal{R}$ . Hence  $\partial\mathcal{R} \subseteq M$ .

**(b) (i)** From Figure 4.12, it appears that the point  $c = \frac{1}{4} + \frac{1}{2}i$  lies on the cardioid with parametrization

$$\gamma(t) = \frac{1}{2}e^{it} - \frac{1}{4}e^{2it} \quad (t \in [-\pi, \pi]).$$

To verify this, either note that

$$\gamma(\pi/2) = \frac{1}{2}i - \frac{1}{4}(-1) = \frac{1}{4} + \frac{1}{2}i,$$

or check that  $c$  satisfies

$$\left(8|c|^2 - \frac{3}{2}\right)^2 + 8\operatorname{Re} c = 3.$$

Thus  $c$  lies on the boundary of the periodic region in  $M$  given by Theorem 4.4(a), and hence  $c \in M$  by part (a).

**(ii)** The point  $c = -1 + \frac{1}{4}i$  lies on the circle  $|c+1| = \frac{1}{4}$ . Thus  $c$  lies on the boundary of the periodic region in  $M$  given by Theorem 4.4(b), and hence  $c \in M$  by part (a).

**4.6** By Theorem 4.5, the function  $P_c$  has a super-attracting 4-cycle if and only if

$$P_c^4(0) = 0, \text{ but } P_c(0), P_c^2(0) \text{ and } P_c^3(0) \text{ are non-zero.}$$

Now  $P_c(0) = c$ ,  $P_c^2(0) = c^2 + c$ ,  $P_c^3(0) = c^4 + 2c^3 + c^2 + c$  and  $P_c^4(0)$  is of the form

$$P_c^4(0) = c^8 + 4c^7 + \dots + c^2 + c.$$

If  $P_c(0) = 0$  or  $P_c^2(0) = 0$ , then  $P_c^4(0) = P_c^2(P_c^2(0)) = 0$ , and so  $P_c^4(0)$  can be factorized as follows:

$$P_c^4(0) = (c^2 + c)(c^6 + 3c^5 + \dots + 1).$$

Thus the values of  $c$  for which  $P_c$  has a super-attracting 4-cycle are the solutions of the equation

$$c^6 + 3c^5 + \dots + 1 = 0,$$

and there are at most six such (distinct) solutions.

Hence, there are at most six values of  $c$  for which  $P_c$  has a super-attracting 4-cycle.

*Remark* In fact it can be shown that each of the polynomial functions  $c \mapsto P_c^n(0)$  has only simple zeros. This makes it possible to count the number of values of  $c$  for which  $P_c$  has a super-attracting  $p$ -cycle, for each positive integer  $p$ . In particular, there are exactly six values of  $c$  for which  $P_c$  has a super-attracting 4-cycle, as indicated in Figure 4.8.

**4.7** For  $c = -\frac{5}{4}$ ,  $P_c$  has the 2-cycle

$$\alpha_1 = -\frac{1}{2} + \sqrt{\frac{1}{2}}, \quad \alpha_2 = -\frac{1}{2} - \sqrt{\frac{1}{2}},$$

(by Problem 2.7(c)) with multiplier

$$\begin{aligned} (P_{-5/4}^2)'(\alpha_1) &= 4\alpha_1\alpha_2 = 4\left(-\frac{1}{2} + \sqrt{\frac{1}{2}}\right)\left(-\frac{1}{2} - \sqrt{\frac{1}{2}}\right) \\ &= -1. \end{aligned}$$

Since  $-1$  is a primitive square root of unity, we deduce, by Theorem 4.6(b), that a period-doubling bifurcation occurs at  $c = -\frac{5}{4}$ .

In Figure 4.8, we see that the point  $-\frac{5}{4}$  lies where a periodic region with period 2 and a periodic region with period 4 touch, so a period-doubling bifurcation is visible.